The inverse of the Abel transform on $\mathbf{SU}^\star(2n)/\mathbf{Sp}(n)$

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Abstract

In this note, we study the inverse of the Abel transform for the symmetric space $\mathbf{SU}^\star(2n)/\mathbf{Sp}(n)$. We start by giving a recursive formula for the dual of the Abel transform on the root system A_{n-1}. This formula allows us to consider a transmutation property on the generalized Abel transform associated to A_{n-1} (see [4] for the root system A_2). We obtain a recursive formula for the transform $\mathcal{A}^{-1}$ on $\mathbf{SU}^\star(2n)/\mathbf{Sp}(n)$. From there, for $n=2$, 3 and 4, we give $\mathcal{A}^{-1}$ as a differential operator in the right coordinates (not the z-coordinates !). The heat kernel of these spaces has a particularly simple form:

$$P_t(x) = C\, e^{-\|\rho\|^2 t}\, t^{-N/2}\, e^{-r^2/(4t)} \sum_{i=0}^{M} \phi_{M-i}(x)\, t^i$$

where $N = \dim\, \mathbf{SU}^\star(2n)/\mathbf{Sp}(n)$, $\phi_0(x)$ is the Legendre function and M is known.

We will discuss these results in relation to a conjecture by Jean-Philippe Anker concerning the behavior of the heat kernel of symmetric spaces of noncompact type.

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0 $\mathbf{SU}^\star(2n)/\mathbf{Sp}(n)$ as a space of positive definite matrices

$\mathbf{SU}^\star(2n)/\mathbf{Sp}(n)$ can be realized as the space of positive definite matrices of determinant 1 over the quaternions $\mathbf{H}$: this comes mainly from the representation of a quaternion number as an ordered

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couple of complex numbers. We will consider instead the space of positive definite matrices over $\mathbf{H}$ without the normalization: this corresponds to $\mathbf{R} \times \mathbf{SU}^\star(2n)/\mathbf{Sp}(n)$ (the first component can be thought of as the log of the determinant). We will denote this space as $\mathbf{Pos}(n,\mathbf{H})$. The heat kernel of $\mathbf{Pos}_1(n,\mathbf{H})$ is easily obtained from that of $\mathbf{Pos}(n,\mathbf{H})$ and vice versa.

The symmetric spaces of noncompact type corresponding to the root system A_{n-1} are $\mathbf{Pos}_1(n,\mathbf{F})$ where $\mathbf{F}$ is $\mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$ (in addition, for the root system A_2, we have $E_{6(-26)}/F_4$).

1 The Abel transform and its inverse

1.1 The Abel transform and its dual

Let $\mathcal{A}(f;e^H) = e^{\rho(H)} \int_N f(e^H n)\, dn$ be the Abel transform of f (also denoted $F_f(e^H)$ in [9]).

It is natural, using the usual Hilbert space structure on L^2 spaces, to define for Weyl-invariant functions, the dual of the Abel transform (see [11] and [15]):

$$\langle \mathcal{A}(h;\cdot), f\rangle_{L^2(A/W)} \quad = \quad \langle h, \mathcal{A}^*(f;\cdot)\rangle_{L^2(K\backslash G/K)}.$$

Using the definition of $\mathcal{A}$ and the integral formulas corresponding to the Iwasawa and the Cartan decompositions (refer to [9]), we find that $\mathcal{A}^*(f;e^H) = \int_K e^{-\rho(H(e^H k))} f(e^{H(e^H k)})\, dk$. In particular, the spherical functions are $\phi_\lambda = \mathcal{A}^*(e^{i\lambda};\cdot)$. This is valid for any symmetric space of noncompact type G/K.

For all $h \in C_c^\infty(K\backslash G/K)$, one has $\mathcal{A}(\Delta h;\cdot) = \Gamma(\Delta)\, \mathcal{A}(h;\cdot)$ where $\Gamma(\Delta) = L_A - \|\rho\|^2$; L_A being the Laplacian on A. The corresponding property for the dual is $\Delta \mathcal{A}^*(f;\cdot) = \mathcal{A}^*(\Gamma(\Delta)\, f;\cdot)$ if f is a smooth Weyl invariant function.

We are mainly interested here by the symmetric spaces of noncompact type that correspond to the root system A_{n-1}.

We express here in a recursive form, the dual of the Abel transform for these spaces:

Theorem 1.1 *We consider the spaces* $\mathbf{Pos}(n, \mathbf{F})$ *where* $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$ *or* $\mathbf{H}$. *Let* $(\mathcal{A}^m)^*(n, f; e^H)$ $(m = \dim_{\mathbf{R}} \mathbf{F})$ *be the dual of the Abel transform applied to a function* f *defined on* A*, the set of* $n \times n$ *diagonal matrices with positive entries. Then, if* $H \in \mathbf{a}^+$*, the diagonal matrices with strictly decreasing entries,*

$$
\begin{aligned}
(\mathcal{A}^m)^*(1, f; e^H) &= f(e^H) \qquad \text{and, for } n \geq 2, \\
(\mathcal{A}^m)^*(n, f; e^H) &= \frac{\Gamma(mn/2)}{(\Gamma(m/2))^n} (d(H))^{1-m} \int_{H_n}^{H_{n-1}} \cdots \int_{H_3}^{H_2} \int_{H_2}^{H_1} (\mathcal{A}^m)^*(n-1, f_{\operatorname{tr} H}; e^{\xi}) \\
&\qquad \cdot [\pm \prod_{i=1}^{n-1} \prod_{j=1}^{n} \sinh(\xi_i - H_j)]^{m/2-1} \, d(\xi) \, d\xi
\end{aligned}
$$

where $d(H) = \prod_{i<j} \sinh(H_i - H_j)$, $\xi = \operatorname{diag}[\xi_1, \ldots, \xi_{n-1}]$, tr *represents the trace and* $f_{\operatorname{tr} H}(e^{\xi}) = f(\exp(\operatorname{diag}[\xi_1, \ldots, \xi_{n-1}, \operatorname{tr} H - \operatorname{tr} \xi]))$. $\pm$ *is chosen so that* $\pm \prod_{i=1}^{n-1} \prod_{j=1}^{n} \sinh(\xi_i - H_j) \geq 0$ *whenever* $H_{i+1} \leq \xi_i \leq H_i$ *for all* i.

Proof: We adapt here an idea of I. M. Gelfand and M. A. Naimark ([7]), by using induction on the integration over K, the maximal compact subgroup of group $\mathbf{GL}_0(n, \mathbf{F})$.

We will denote by K_1 the maximal compact subgroup of $\mathbf{GL}_0(n-1, \mathbf{F})$. K_1 is a subgroup of K when identified with those elements of K that fixes $\mathbf{e}_n = [0, \ldots, 0, 1]^T$. Now,

$$\int_K f(k) \, dk = \int_{K/K_1} \int_{K_1} f(\underline{k} k_1) \, dk_1 \, d\underline{k} = \int_{\mathbf{S}^{mn-1}} [\int_{K_1} f([J(\mathbf{x}) k_1; \mathbf{x}]) \, dk_1] \, d\nu(\mathbf{x}).$$

$[J(\mathbf{x}); \mathbf{x}]$ belongs to K; it is easy to show that $J(\mathbf{x})$ can be made to depend smoothly on $\mathbf{x}$ in a dense subset of $\mathbf{S}^{mn-1} = \{\mathbf{x} \in \mathbf{F}^n : |\mathbf{x}| = 1\}$.

If $g = ke^{H(g)}n$ (the Iwasawa decomposition), then

$$e^{2H(g)} = \mathrm{diag}\,[\Delta_p(g^*g)/\Delta_{p-1}(g^*g)]_{1\le p\le n} \tag{1}$$

where $\Delta_p(Q) = \det(Q_{ij})_{1\le i,j\le p}$ for $1 \le p \le n$ (the **Gram determinant** of Q for the first p rows and columns) and $\Delta_0(Q) = 1$. T. S. Bhanu Murti uses the Gram determinant explicitly in [13] to describe the Plancherel measure on $\mathbf{SL}(n,\mathbf{R})/\mathbf{SO}(n)$.

In what follows, Q will be a positive definite matrix with eigenvalues $\lambda_1 > \lambda_2 > \ldots > \lambda_n > 0$.

The main ingredient of the proof is the computation of the "I-transform"

$$\begin{aligned} I(n,f;Q) &= \int_K f((\Delta_p(k^*Qk))_{1\le p\le n})\,dk \\ &= \int_{\mathbf{S}^{mn-1}} [\int_{K_1} f((\Delta_p([J(\mathbf{x})k_1;x]^*Q[J(\mathbf{x})k_1;\mathbf{x}]))_{1\le p\le n})\,dk_1\,]\,d\nu(\mathbf{x}) \end{aligned} \tag{2}$$

for a continuous function $f : (0,\infty)^n \to \mathbf{C}$. It is clear that $I(n,f;Q)$ depends only on the eigenvalues on Q in any order.

We will show that the inner integral of (2) corresponds to an "I-transform" with parameter $n-1$ and argument $J(\mathbf{x})^*QJ(\mathbf{x})$, and that the outer integral can be parametrized by the eigenvalues of $J(\mathbf{x})^*QJ(\mathbf{x})$.

Observe that $J(\mathbf{x}) : \mathbf{F}^{n-1} \to \mathbf{x}^\perp \subset \mathbf{F}^n$ is one to one and onto, $J(\mathbf{x})^* : \mathbf{F}^n \to \mathbf{F}^{n-1}$ is onto with $\ker J(\mathbf{x})^* = \{\iota\,\mathbf{x} : \iota \in \mathbf{F}\}$ and $J(\mathbf{x})^*J(\mathbf{x}) = I_{n-1}$. $J(\mathbf{x})^*QJ(\mathbf{x})$ is clearly positive definite.

$$\Delta_p([J(\mathbf{x})k_1;\mathbf{x}]^*Q[J(\mathbf{x})k_1;\mathbf{x}]) = \begin{cases} \Delta_p(k_1^*J(\mathbf{x})^*QJ(\mathbf{x})k_1) & \text{if } 1\le p\le n-1, \\ \det Q & \text{if } p = n. \end{cases} \tag{3}$$

It is clear now, that the inner integral of (2) corresponds to $I(n-1, f_{\det Q}; J(\mathbf{x})^*QJ(\mathbf{x}))$ if $f_{\det Q}(a_1,\ldots,a_{n-1}) = f(a_1,\ldots,a_{n-1},\det Q)$. Our goal is then, for appropriate functions, to express the measure $d\nu(\mathbf{x})$ on $\mathbf{S}^{mn-1}$ in terms of the of the eigenvalues of $J(\mathbf{x})^*QJ(\mathbf{x})$.

We choose $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_n\}$ an orthonormal basis of $\mathbf{F}^n$ for which $Q\mathbf{v}_s = \lambda_s\mathbf{v}_s$. We write $\mathbf{x} = \sum_{s=1}^n x_s\mathbf{v}_s$ and assume that $x_s \neq 0$ for all s (we are neglecting a set of measure 0).

Let $\mathbf{w}$ be an eigenvector of $J(\mathbf{x})^*QJ(\mathbf{x})$ with eigenvalue μ; $J(\mathbf{x})^*QJ(\mathbf{x})\mathbf{w} = \mu\mathbf{w}$ hence $QJ(\mathbf{x})\mathbf{w} = \mu J(\mathbf{x})\mathbf{w} + \iota\mathbf{x}$ (it is easy to see that $QJ(\mathbf{x})\mathbf{w} - \mu J(\mathbf{x})\mathbf{w}$ is perpendicular to $J(\mathbf{x})\,\mathbf{F}^{n-1} = \mathbf{x}^\perp$).

$J(\mathbf{x})\mathbf{w}$ is not an eigenvector of Q. Otherwise $J(\mathbf{x})\mathbf{w}$ would be a multiple of one of the eigenvectors $\mathbf{v}_s$ and would not be perpendicular to $\mathbf{x}$ since $x_s \neq 0$ for all s. As a consequence, $\iota \neq 0$.

We write $J(\mathbf{x})\mathbf{w} = \sum_{s=1}^n \iota_s\mathbf{v}_s$. The relation $QJ(\mathbf{x})\mathbf{w} = \mu J(\mathbf{x})\mathbf{w} + \iota\mathbf{x}$ gives us $\sum_{s=1}^n \lambda_s\iota_s\mathbf{v}_s = \sum_{s=1}^n \mu\iota_s\mathbf{v}_s + \sum_{s=1}^n \iota x_s\mathbf{v}_s$ so $\lambda_s\iota_s = \mu\iota_s + \iota x_s$ and hence, $\iota_s = \iota\,\frac{x_s}{\lambda_s-\mu}$ ($\mu = \lambda_s$ would imply $x_s = 0$ or $\iota = 0$). The eigenvalue μ determines $\mathbf{w}$ up to a constant factor since $\mathbf{w} = J(\mathbf{x})^*J(\mathbf{x})\mathbf{w} = \iota\sum_{s=1}^n \frac{x_s}{\lambda_s-\mu}\,J(\mathbf{x})^*\mathbf{v}_s$. In other words, $J(\mathbf{x})^*QJ(\mathbf{x})$ has $n-1$ distinct eigenvalues.

Now, $\langle J(\mathbf{x})\mathbf{w}, \mathbf{x}\rangle = 0$ implies $0 = \langle\sum_{s=1}^n \iota_s\mathbf{v}_s, \sum_{s=1}^n x_s\mathbf{v}_s\rangle = \sum_{s=1}^n \iota_s\overline{x_s}$ so $\iota\sum_{s=1}^n \frac{|x_s|^2}{\lambda_s-\mu} = 0$. If we write $t_s = |x_s|^2$, we have $\sum_{s=1}^n \frac{t_s}{\lambda_s-\mu} = 0$ and $\sum_{s=1}^n t_s = 1$ with $t_s > 0$.

These equations imply that each eigenvalue μ is squeezed between two eigenvalues of Q (otherwise $\sum_{s=1}^n \frac{t_s}{\lambda_s-\mu}$ would be strictly positive or strictly negative).

Let $\mu_1 > \ldots > \mu_{n-1}$ be the eigenvalues of $J(\mathbf{x})^*QJ(\mathbf{x})$. If $i < j$, then $0 = \sum_{s=1}^n \frac{t_s}{\lambda_s-\mu_i} - \sum_{s=1}^n \frac{t_s}{\lambda_s-\mu_j} = (\mu_i - \mu_j)\sum_{s=1}^n \frac{t_s}{(\lambda_s-\mu_i)(\lambda_s-\mu_j)}$, which means that we cannot have $i < j$ and $\lambda_r > \mu_i > \mu_j > \lambda_{r+1}$. We then have $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \ldots > \lambda_{n-1} > \mu_{n-1} > \lambda_n$.

Let $\nu_i = \frac{1}{\mu_i}$, $\nu_n = 0$: $\sum_{k=1}^n \frac{t_s}{1-\nu_i\lambda_s} = \begin{cases} 0 & \text{if } 1 \le i \le n-1, \\ 1 & \text{if } i = n. \end{cases}$

Using the formula $\det\left[\frac{1}{1-a_i\,b_k}\right]_{i,k=1,\ldots,n} = \frac{\prod_{i>k}(a_i-a_k)\prod_{i>k}(b_i-b_k)}{\prod_{i=1}^n\prod_{k=1}^n(1-a_ib_k)}$ (see [16, page 202]) and Cramer's rule, we find: $t_p = \frac{\prod_{i=1}^{n-1}(\mu_i-\lambda_p)}{\prod_{i\neq p}(\lambda_i-\lambda_p)}$.

At last, using straightforward calculus, $\frac{D(t_i)}{D(\mu_j)} = \frac{\prod_{i<p}(\mu_i-\mu_p)}{\prod_{i<p}(\lambda_i-\lambda_p)}$.

The coordinates $\{x_s\}$ provide a parametrization for $\mathbf{S}^{mn-1}$ as long as the basis $\{\mathbf{v}_s\}$ is fixed. For functions which depends only on $t_s = |x_s|^2$ and with the normalization $\int_{\mathbf{S}^{mn-1}} d\nu(\mathbf{x}) = 1$:

$$\begin{aligned} d\nu(\mathbf{x}) &= \frac{\Gamma(mn/2)}{(\Gamma(m/2))^n}\,(t_1\ldots t_n)^{m/2-1}\,dt_1\ldots dt_{n-1} \\ &= \frac{\Gamma(mn/2)}{(\Gamma(m/2))^n}\prod_{p=1}^{n}\left[\frac{\prod_{i=1}^{n-1}|\mu_i-\lambda_p|}{\prod_{i\neq p}|\lambda_i-\lambda_p|}\right]^{m/2-1}\frac{\prod_{i<p}(\mu_i-\mu_p)}{\prod_{i<p}(\lambda_i-\lambda_p)}\,d\mu_1\ldots d\mu_{n-1} \\ &= \frac{\Gamma(mn/2)}{(\Gamma(m/2))^n}\prod_{i<j}(\lambda_i-\lambda_j)^{1-m}\prod_{i=1}^{n-1}\prod_{j=1}^{n}|\mu_i-\lambda_j|^{m/2-1}\prod_{i<p}(\mu_i-\mu_p)\,d\mu_1\ldots d\mu_{n-1} \end{aligned}$$

We then have

$$I(n,f;Q) = \frac{\Gamma(mn/2)}{(\Gamma(m/2))^n}\int_{\lambda_n}^{\lambda_{n-1}}\cdots\int_{\lambda_2}^{\lambda_1} I(n-1,f_{\det Q};\mathrm{diag}\,[\mu_1,\ldots,\mu_{n-1}])\,\Psi(\lambda,\mu)\,d\mu_1\ldots d\mu_{n-1}$$

where $f_{\det Q}(a_1,\ldots,a_{n-1}) = f(a_1,\ldots,a_{n-1},\det Q/\prod_{i=1}^{n-1} a_i)$ and

$$\Psi(\lambda,\mu) = [\prod_{i<j}(\lambda_i-\lambda_j)]^{1-m}\prod_{i=1}^{n-1}\prod_{j=1}^{n}|\mu_i-\lambda_j|^{m/2-1}\prod_{i<p}(\mu_i-\mu_p).$$

If we refer to equation (1), it is plain that $(\mathcal{A}^m)^*(n,f;e^H)$ can be expressed as an I-transform. With the change of variables $\mu_i = e^{2\xi_i}$ $(1\le i\le n-1)$, the remainder of the proof is straightforward.

∎

Theorem 1.1 can be found in the author's McGill University doctoral dissertation [14]. Although it is not proven there, it provided the "inspiration" in finding the inverse of the Abel transform for the spaces $\mathbf{Pos}(n,\mathbf{R})$ in [15].

In [4], R. J. Beerends discusses a generalized Abel transform on the root system A_{n-1} where the multiplicity of the roots is allowed to be a complex number (which we will denote m). We will follow the notation in [4].

Theorem 1.1 permits us to discuss the generalized Abel transform (or rather its dual) on the root system A_{n-1} as long as $\Re m > 0$.

$\rho(m) = \frac{1}{2}\sum_{i<j} m\,(H_i - H_j)$ and the radial part of the generalized Laplace-Beltrami operator is defined as

$$L(m) = \sum_{j=1}^{n} \frac{\partial^2}{\partial H_j^2} + m \sum_{j=1}^{n} \sum_{k \neq j} \coth(H_j - H_k)\, \frac{\partial}{\partial H_j}.$$

Definition 1.2 *Let us fix m ($\Re m > 0$). For $H \in \mathbf{a}^+$, the diagonal matrices with strictly decreasing entries, we can define the dual of the generalized Abel transform $\mathcal{A}^m$ for the root system on A_{n-1}:*

$$\begin{aligned}
(\mathcal{A}^m)^*(1, f; e^H) &= f(e^H) \qquad \text{and, for } n \geq 2,\\
(\mathcal{A}^m)^*(n, f; e^H) &= \frac{\Gamma(mn/2)}{(\Gamma(m/2))^n}\,(d(H))^{1-m} \int_{H_n}^{H_{n-1}} \cdots \int_{H_3}^{H_2} \int_{H_2}^{H_1} (\mathcal{A}^m)^*(n-1, f_{\mathrm{tr}\,H}; e^{\xi})\\
&\qquad \cdot [\pm \prod_{i=1}^{n-1} \prod_{j=1}^{n} \sinh(\xi_i - H_j)]^{m/2-1}\, d(\xi)\, d\xi
\end{aligned}$$

where $\pm$ is chosen so that $\pm \prod_{i=1}^{n-1} \prod_{j=1}^{n} \sinh(\xi_i - H_j) \geq 0$ whenever $H_{i+1} \leq \xi_i \leq H_i$ for all i.

Theorem 1.3 *Suppose f is a smooth Weyl invariant function. Then, $(\mathcal{A}^m)^*(n, f; \cdot)$ is smooth, Weyl invariant and*

$$L(m)\,(\mathcal{A}^m)^*(n, f; \cdot) \;=\; (\mathcal{A}^m)^*(n, \Gamma(L(m))f; \cdot)$$

$(\Gamma(L(m)) = L_A - \|\rho(m)\|^2\rangle)$.

Proof: The result can be proven by induction on n. The idea is to take $\Re m$ large enough in order to use integration by parts without adding new terms. By analytic continuation, the result is true for $\Re m > 0$. The proof is similar, but simpler, to that of Theorem 2.5 in [15]. ∎

1.2 The inverse of the Abel on a symmetric space for which the roots all have even multiplicities

The Harish-Chandra c-function for such a space has a rather simple form (see [9]):

$$\mathbf{c}(\lambda)^{-1} = c_0 \prod_{\eta\in\Sigma^+} \left[\left(\frac{\langle i\lambda, \eta\rangle}{\langle \eta, \eta\rangle} \right)_{m_\eta/2} \right]$$

where c_0 is a constant and $(u)_p = \prod_{i=0}^{p-1} (u+i)$.

Let $\mathcal{Z}$ be the Weyl invariant differential operator with constant coefficients such that $\mathcal{Z}(e^{i\lambda(H)}) = |c(\lambda)|^{-2}\, e^{i\lambda(H)}$ for $\lambda \in \mathbf{a}^*$, the set of real-valued linear functionals on $\mathbf{a}$. If $\tilde{f}$ is the spherical transform of the K-invariant function f then, under suitable conditions for f,

$$\begin{aligned}
\tilde{f}(\lambda) &= \int_{\mathbf{a}} e^{-i\lambda(H)}\, \mathcal{A}(f)(e^H)\, dH \\
\mathcal{A}(f)(e^H) &= \int_{\mathbf{a}^*} \tilde{f}(\lambda)\, e^{i\lambda(H)}\, d\lambda \\
\mathcal{Z}(\mathcal{A}(f))(e^H) &= \int_{\mathbf{a}^*} \tilde{f}(\lambda)\, e^{i\lambda(H)}\, |c(\lambda)|^{-2}\, d\lambda \\
\mathcal{A}^*(\mathcal{Z}(\mathcal{A}(f)))(e^H) &= \int_{\mathbf{a}^*} \tilde{f}(\lambda)\, \phi_\lambda(e^H)\, |c(\lambda)|^{-2}\, d\lambda = C\, f(e^H).
\end{aligned}$$

(in this note, C will always stand for a strictly positive constant whose value may change from equation to equation).

This brings:

Theorem 1.4 *The inverse of the Abel on a symmetric space of noncompact type for which the roots all have even multiplicities is given by*

$$\mathcal{A}^{-1} = C^{-1}\, \mathcal{A}^*\, \mathcal{Z}$$

where $\mathcal{Z}$ is the differential operator with constant coefficients such that $\mathcal{Z}(e^{i\lambda(H)}) = |c(\lambda)|^{-2}\, e^{i\lambda(H)}$ for $\lambda \in \mathbf{a}^*$.

These considerations together with Theorem 1.1 allow for a recursive formula for the inverse of the Abel transform for the space $\mathbf{Pos}(n, \mathbf{H})$.

Theorem 1.5 *By $\mathcal{A}^{-1}(n, f; \cdot)$, we will understand the inverse of the Abel transform for $\mathbf{Pos}(n, \mathbf{H})$. If f is a smooth compactly supported function on A and if $H \in \mathbf{a}^+$, then*

$$\begin{aligned} \mathcal{A}^{-1}(1, f; e^H) &= f(e^H) \qquad \text{and, for } n \geq 2, \\ \mathcal{A}^{-1}(n, f; e^H) &= C_n\, (d(H))^{-3} \int_{H_n}^{H_{n-1}} \cdots \int_{H_3}^{H_2} \int_{H_2}^{H_1} \mathcal{A}^{-1}\left(n-1, \prod_{i=1}^{n-1} P(\frac{\partial}{\partial \xi_i})\, f_{\operatorname{tr} H}; e^{\xi}\right) \\ &\quad \cdot \prod_{i=1}^{n-1} \prod_{j=1}^{n} \sinh(\xi_i - H_j)\, d(\xi)\, d\xi \end{aligned}$$

where $P(z) = z^4 - 4z^2$.

Proof: The present result follows from Theorems 1.4 and 1.1. Note that in this case, $\mathcal{Z} = 2^{-2n(n-1)}\, c_0^2 \prod_{i<j\leq n} \left(\left(\frac{\partial}{\partial H_i} - \frac{\partial}{\partial H_j} \right)^4 - 4 \left(\frac{\partial}{\partial H_i} - \frac{\partial}{\partial H_j} \right)^2 \right)$. ∎

We obtained a similar result for the space $\mathbf{Pos}(n, \mathbf{R})$ in [15] by a much more involved proof. The inverse of the Abel transform for the space $\mathbf{Pos}(n, \mathbf{C})$ is simply $C\, \frac{1}{d}\, \partial(\pi)$ where $\partial(\pi) = \prod_{i<j} \left(\frac{\partial}{\partial H_i} - \frac{\partial}{\partial H_j} \right)$.

2 $\mathbf{Pos}(2, \mathbf{H})$

Theorem 2.1 *$\mathcal{A}^{-1} = C\, \frac{1}{d}\, \partial(\pi) \circ \frac{1}{d}\, \partial(\pi)$ is the inverse of the Abel transform for the space $\mathbf{Pos}(2, \mathbf{H})$ ($\alpha = H_1 - H_2$ and $\beta = H_2 - H_3$).*

Proof: It suffices to use Theorem 1.5 and integration by parts. We actually show that $\mathcal{A}^* \left(2, \tilde{P}(\partial_\alpha)\, g; e^H\right) = C\, \frac{1}{d}\, \partial(\pi) \circ \frac{1}{d}\, g$ if g is an odd function ($g(\exp(s \cdot H)) = (\det s)\, g(e^H)$ for s in the Weyl group) and $\tilde{P}(z) = z^3 - 4z$. ∎

Corollary 2.2 *The heat kernel for the space* $\mathbf{Pos}(2, \mathbf{H})$ *is given by*

$$\begin{aligned} P_t(e^H) &= C\, d^{-3}(H) \int_{H_2}^{H_1} P(\frac{\partial}{\partial \xi}) \left\{ e^{-\|\rho\|^2 t}\, t^{-1} \exp(-r^2(F)/(4t)) \right\}_{F_1=\xi,\ F_2=H_1+H_2-\xi} \\ &\qquad\qquad \cdot \sinh(\xi - H_1) \sinh(\xi - H_2)\, d\xi \\ &= C \left(\frac{1}{d}\, \partial(\pi) \right)^2 \left[e^{-\|\rho\|^2 t}\, t^{-2/2}\, e^{-r^2/(4t)} \right] \end{aligned}$$

where $r^2 = r^2(F) = F_1^2 + F_2^2$*,* P *is as before and, in this case,* $d = \sinh \alpha$*.*

Proof: It suffices to use Theorems 1.5, 2.1 and integration by parts. The Abel transform of the heat kernel for a symmetric space of noncompact type is known to be $C\, e^{-\|\rho\|^2 t}\, t^{-l/2}\, e^{-r^2/(4t)}$ where l is the rank of the space (see [12]). ∎

Note that in [10], Carl S. Herz gives an alternate description of the heat kernel as

$$\begin{aligned} P_t(e^H) &= C\, d^{-3}(H) \int_{H_1}^{\infty} P(\frac{\partial}{\partial \xi}) \left\{ e^{-\|\rho\|^2 t}\, t^{-1} \exp(-r^2(F)/(4t)) \right\}_{F_1=\xi,\ F_2=H_1+H_2-\xi} \\ &\qquad\qquad \cdot \sinh(\xi - H_1) \sinh(\xi - H_2)\, d\xi \end{aligned}$$

(taking into account the normalization which is different).

We recall that $\mathbf{Pos}_1(2, \mathbf{H})$ is the same as $H^{2 \cdot 2+1}$. It can be shown that for the spaces H^{2n+1}, we have $\mathcal{A}^{-1} = C_n \left(\frac{1}{\sinh \alpha} \frac{\partial}{\partial \alpha} \right)^n$.

3 $\mathbf{Pos}(3, \mathbf{H})$

Theorem 3.1 *If* f *is smooth and Weyl invariant, one finds that*

$$\mathcal{A}^{-1}(3, f, \cdot) = C\, \frac{1}{d^3}\, \mathbf{D}\, \partial(\pi)\, f$$

where

Definition 3.2

$$
\begin{aligned}
\mathbf{D} \;=\;& 4\,(\cosh^2\alpha + \cosh^2\beta + \cosh^2(\alpha+\beta)) \\
& - 2\,\sinh(2\alpha)\,\partial_\alpha - 2\,\sinh(2\beta)\,\partial_\beta - 2\,\sinh(2(\alpha+\beta))\,\partial_{\alpha+\beta} \\
& + (3\,\sinh^2(\alpha+\beta) - \sinh^2\alpha - \sinh^2\beta)\,\partial_\alpha\,\partial_\beta \\
& + (\sinh^2\alpha + \sinh^2(\alpha+\beta) - \sinh^2\beta)\,\partial_\alpha^2 \\
& + (\sinh^2\beta + \sinh^2(\alpha+\beta) - \sinh^2\alpha)\,\partial_\beta^2 \\
& - d(H)\,\partial(\pi).
\end{aligned}
$$

Proof: We will first compute $d^3(H)\,\mathcal{A}^*(2, \tilde{P}(\partial_\alpha)\,\tilde{P}(\partial_\beta)\,\tilde{P}(\partial_{\alpha+\beta})\,g; e^H)$ where, as before, $\tilde{P}(z) = z^3 - 4z$ and g is a smooth odd function which will eventually be replaced by $\partial(\pi)\,f$. We need to refer to the proof of Theorem 2.1 and use integration by parts.

$$
\begin{aligned}
& d^3(H)\,\mathcal{A}^*(3, \tilde{P}(\partial_\alpha)\,\tilde{P}(\partial_\beta)\,\tilde{P}(\partial_{\alpha+\beta})\,g; e^H) \\
=\;& \int_{H_3}^{H_2}\int_{H_2}^{H_1} \mathcal{A}^*(2, \Big(\tilde{P}(\partial_\alpha)\,\tilde{P}(\partial_\beta)\,\tilde{P}(\partial_{\alpha+\beta})\,g\Big)_{\mathrm{tr}\,H}; e^\xi)\prod_{i=1}^{2}\prod_{j=1}^{3}\sinh(\xi_i - H_j)\;\sinh(\xi_1-\xi_2)\,d\xi_1\,d\xi_2 \\
=\;& \int_{H_3}^{H_2}\int_{H_2}^{H_1} \frac{1}{\sinh(\xi_1-\xi_2)}\left(\frac{\partial}{\partial\xi_1} - \frac{\partial}{\partial\xi_2}\right)\circ\frac{1}{\sinh(\xi_1-\xi_2)}\,(\tilde{P}(\partial_{\alpha+\beta})\tilde{P}(\partial_\beta)g)_{\mathrm{tr}\,H}(e^\xi) \\
& \qquad\cdot\prod_{i=1}^{2}\prod_{j=1}^{3}\sinh(\xi_i - H_j)\;\sinh(\xi_1-\xi_2)\,d\xi_1\,d\xi_2 \\
=\;& \int_{H_3}^{H_2}\int_{H_2}^{H_1} \left(\frac{\partial}{\partial\xi_1} - \frac{\partial}{\partial\xi_2}\right)\circ\frac{1}{\sinh(\xi_1-\xi_2)}\,(\tilde{P}(\partial_{\alpha+\beta})\,\tilde{P}(\partial_\beta)g)_{\mathrm{tr}\,H}(e^\xi)\prod_{i=1}^{2}\prod_{j=1}^{3}\sinh(\xi_i - H_j)\,d\xi_1\,d\xi_2 \\
=\;& \sum_{k=1}^{3}\int_{H_3}^{H_2}\int_{H_2}^{H_1} (\tilde{P}(\partial_{\alpha+\beta})\,\tilde{P}(\partial_\beta)g)_{\mathrm{tr}\,H}(e^\xi)\prod_{i=1}^{2}\prod_{j\neq k}\sinh(\xi_i - H_j)\,d\xi_1\,d\xi_2 \\
=\;& \sum_{k=1}^{3}\int_{H_3}^{H_2}\int_{H_2}^{H_1} \tilde{P}(\frac{\partial}{\partial\xi_1})\,\tilde{P}(\frac{\partial}{\partial\xi_2})\,g_{\mathrm{tr}\,H}(e^\xi)\prod_{i=1}^{2}\prod_{j\neq k}\sinh(\xi_i - H_j)\,d\xi_1\,d\xi_2
\end{aligned}
$$

$$= \sum_{k=1}^{3} \left[Q(\frac{\partial}{\partial \xi_2}, H_2, H_i, H_j)\Big|_{\xi_2=H_2} - Q(\frac{\partial}{\partial \xi_2}, H_3, H_i, H_j)\Big|_{\xi_2=H_3} \right]$$

$$\cdot \left[Q(\frac{\partial}{\partial \xi_1}, H_1, H_i, H_j)\Big|_{\xi_1=H_1} - Q(\frac{\partial}{\partial \xi_1}, H_2, H_i, H_j)\Big|_{\xi_1=H_2} \right] g_{\mathrm{tr}\, H}(e^x i) \tag{4}$$

$$(i < j,\ i \neq k \text{ and } j \neq k)$$

where $Q(D, a, u_1, u_2) = \sinh(a - u_1)\,\sinh(a - u_2)\,D^2 - \sinh(2a - u_1 - u_2)\,D + 2\,\cosh(u_1 - u_2)$.

To simplify (4) we need the following: for g odd,

$$\begin{aligned}
(\partial_\beta^i\,\partial_{\alpha+\beta}^j\,g)(H_2, H_2, H_1 + H_3 - 2H_2) &= -(\partial_\beta^j\,\partial_{\alpha+\beta}^i\,g)(H_2, H_2, H_1 + H_3 - 2H_2),\\
(\partial_\beta^i\,\partial_{\alpha+\beta}^j\,g)(H_1, H_3, H_2) &= -(-1)^i\,(\partial_\alpha^j\,\partial_\beta^i\,g)(H_1, H_2, H_3),\\
(\partial_\beta^i\,\partial_{\alpha+\beta}^j\,g)(H_2, H_3, H_1) &= (-1)^i\,(-1)^j\,(\partial_\alpha^j\,\partial_{\alpha+\beta}^i\,g)(H_1, H_2, H_3).
\end{aligned}$$

The remaining computations are simple but tedious. They were done with the help of the software Mathematica ([17]) on a NeXTstation computer. ∎

As a consequence:

Corollary 3.3

$$\Delta \circ \frac{1}{d^3}\,\mathbf{D} = \frac{1}{d^3}\,\mathbf{D}\,(L_A - \|\rho\|^2).$$

Proof: The fact that the function g used in the proof of Theorem 3.1 was odd poses no problem: the equality of differential operators is a local property. The corollary could also be proven directly.

∎

These results were obtained by A. Hba (in [8]) and R. J. Beerends (see [3]) using different methods.

We want a simpler form for the operator $\frac{1}{d^3}\,\mathbf{D}$.

Definition 3.4 *We define* $\operatorname{sgn}\alpha = -1$, $\operatorname{sgn}\beta = -1$ *and* $\operatorname{sgn}(\alpha+\beta) = 1$ *and*

$$\Box \;=\; \partial(\pi) + \sum_{\eta>0} (\operatorname{sgn}\eta)\, \partial_\eta \circ \coth\eta\, \partial_\eta. \tag{5}$$

Theorem 3.5 $\frac{1}{d^3}\,\mathbf{D} = -\frac{1}{d}\,\Box\circ\frac{1}{d}$.

Proof: Proving the result, using (5), is another tedious computation. It is equivalent but simpler to show that $\mathbf{D}\circ d = -d^2\,\Box$. ∎

The following result is a also consequence of the general theory on the Abel transform.

Theorem 3.6 *If f is smooth and odd then $\frac{1}{d^3}\,\mathbf{D}\,f$ is smooth and Weyl invariant.*

Proof: First note that if f is odd and smooth then $\frac{f}{d}$ is smooth and Weyl invariant. Now for each root η, it is easy to see that $\coth\eta\,\partial_\eta\left(\frac{f}{d}\right)$ is smooth (the singularity $\eta = 0$ is removable). It follows that $\Box\left(\frac{f}{d}\right)$ is smooth and odd (if h is Weyl invariant, then $\Box(h)$ is odd). Finally, this means that $\frac{1}{d}\,\Box\left(\frac{f}{d}\right)$ is smooth and Weyl invariant. ∎

Theorem 3.7

$$P_t(e^H) = C\,\frac{1}{d^3}\,\mathbf{D}\,\partial(\pi)\,\left[e^{-\|\rho\|^2 t}\,t^{-3/2}\,e^{-r^2/(4t)}\right] = C\,e^{-\|\rho\|^2 t}\,t^{-15/2}\,e^{-r^2/(4t)}\sum_{i=0}^{3}\phi_{m-i}(H)\,t^i$$

is the fundamental solution of the heat equation on the symmetric space $\mathbf{Pos}(3,\mathbf{H})$ *for an appropriate choice of the constant C ($r^2 = r^2(H) = \sum_{k=1}^{n} H_k^2$).*

Proof: It suffices to refer to Theorem 3.1. ∎

The functions $\phi_i(e^H)$ can be computed explicitly. The simplest way to do that is to use $\mathbf{D}$ of Definition 3.2 and compute the coefficients $R_i(\alpha,\beta) = d^3(H)\,\phi_i(e^H)$ of t^{3-i} ($i = 0,\,1,\,2,\,3$) in

$e^{\|\rho\|^2 t}\, t^{15/2}\, d^3(H)\, e^{r^2/(4t)}\, P_t(e^H)$. For example:

$$\begin{aligned}
-64\, R_0(\alpha,\beta) &= C\, d^3(H)\, \phi_0(e^H) \\
&= -12\, \sinh\alpha\, \sinh\beta\, \sinh(\alpha+\beta) - 3\,\alpha\,(3\,\sinh^2\alpha - \sinh^2\beta - \sinh^2(\alpha+\beta)) \\
&\quad - 3\,\beta\,(3\,\sinh^2\beta - \sinh^2\alpha - \sinh^2(\alpha+\beta)) \\
&\quad + 3\,(\alpha+\beta)\,(3\,\sinh^2(\alpha+\beta) - \sinh^2\alpha - \sinh^2\beta) - 2\,\sinh(2\alpha)\,(\beta^2 + (\alpha+\beta)^2 - 2\alpha^2) \\
&\quad - 2\,\sinh(2\beta)\,(\alpha^2 + (\alpha+\beta)^2 - 2\beta^2) - 2\,\sinh(2(\alpha+\beta))\,(2(\alpha+\beta)^2 - \alpha^2 - \beta^2) \\
&\quad + 4\,(\cosh^2\alpha + \cosh^2\beta + \cosh^2(\alpha+\beta))\,\alpha\,\beta\,(\alpha+\beta).
\end{aligned}$$

4 The inverse of the Abel transform for $\mathbf{Pos}(n, \mathbf{H})$: another method

In what follows, unless otherwise specified, m is any complex number and the root system under study is A_{n-1}.

The factor d^{1-m} in front of the expression for the dual of the generalized Abel transform in Theorem 1.2 suggests that the following result might prove useful (using the notation of Theorem 1.3):

Lemma 4.1

$$(L(m) + \|\rho(m)\|^2) \circ d^{1-m} = d^{1-m}\,(L(2-m) + \|\rho(2-m)\|^2).$$

Proof: It suffices to compute. ∎

We adapt here the terminology of Chapter IV, §5 in Helgason's [9].

Let Λ be the set of all linear combinations of the positive roots having non-negative integer coefficients.

We want to consider the eigenfunction of $L(m)$ corresponding to the eigenvalue $-(\langle\lambda,\lambda\rangle + \langle\rho(m),\rho(m)\rangle)$ and which has the expansion

$$\Phi_\lambda^m(H) = e^{(i\lambda-\rho(m))(H)} \sum_{\mu\in\Lambda} \Gamma_\mu^m(\lambda)\, e^{-\mu(H)} \tag{6}$$

(under standard restrictions on λ, the expansion exists and is unique once $\Gamma_0^m(\lambda)$ has been chosen).

It is clear from Lemma 4.1 that $d^{1-m}\,\Phi_\lambda^{2-m}$ will be an eigenfunction of $L(m)$ for the eigenvalue $-(\langle\lambda,\lambda\rangle+\langle\rho(m),\rho(m)\rangle)$. Since it has the right expansion, it has to be equal to $\Gamma_0^{2-m}(\lambda)/\Gamma_0^m(\lambda)\,\Phi_\lambda^m$.

If we imitate the procedure used in [9], we find that

$$\{\langle\mu,\mu\rangle - 2\langle i\lambda,\mu\rangle\}\,\Gamma_\mu^m(\lambda) = 2m \sum_{\alpha>0}\sum_{k\geq 1}\{\langle\mu-2k\alpha+\rho(m)-i\lambda,\alpha\rangle\}\,\Gamma_{\mu-2k\alpha}^m(\lambda). \tag{7}$$

It is clear that $\Gamma_\mu^m(\lambda)=0$ if $\mu\notin 2\Lambda$.

In particular, we have $\Phi_\lambda^0 = \Gamma_0^0(\lambda)\,e^{i\lambda}$. This in turn shows that $\Phi_\lambda^2 = \Gamma_0^2(\lambda)\,d^{-1}\,e^{i\lambda}$ which is the well known expression for these functions in the case of the positive definite hermitian matrices.

Observation leads one to expect that the expansion in (6) terminates if $m=-2q$ where q is a non negative integer. In that case, and in the case corresponding to $2-m$, the inverse of the Abel transform can be given explicitly as a differential operator. We have to choose $\Gamma_0^m(\lambda)$ so that all the $\Gamma_\mu^m(\lambda)$ are polynomials. It is then easy to find the differential operator that sends $e^{i\lambda}$ to $\Phi_\lambda^m/c(-\lambda)$. That operator has to be $\mathcal{A}^{-1}$ (see [3]).

We show that if $n=2$ the series terminates when $m=-2q$, q a non-negative integer. Let $\mu=2j\alpha$; (7) becomes

$$\begin{aligned}&(8j^2-4j\langle\alpha,i\lambda\rangle)\,\Gamma_{2j\alpha}^{-2q}(\lambda)\\ &= -4q\sum_{k\geq 1}(4j-4k-2q-\langle\alpha,i\lambda\rangle))\,\Gamma_{2(j-k)\alpha}^{-2q}(\lambda)\end{aligned}$$

$$
\begin{aligned}
&= -4q \sum_{k\geq 2} (4j-4k-2q-\langle\alpha, i\lambda\rangle)\,\Gamma^{-2q}_{2(j-k)\alpha}(\lambda) - 4q\,(4j-4-4q-\langle\alpha, i\lambda\rangle)\,\Gamma^{-2q}_{2(j-1)\alpha}(\lambda) \\
&= 4q \sum_{k\geq 1} (4(j-1)-4k-2q-\langle\alpha, i\lambda\rangle)\,\Gamma^{-2q}_{2(j-1-k)\alpha}(\lambda) - 4q\,(4j-4-2q-\langle\alpha, i\lambda\rangle)\,\Gamma^{-2q}_{2(j-1)\alpha}(\lambda) \\
&= (8(j-1)^2 - 4(j-1)\langle\alpha, i\lambda\rangle)\,\Gamma^{-2q}_{2(j-1)\alpha}(\lambda) - 4q\,(4j-4-2q-\langle\alpha, i\lambda\rangle)\,\Gamma^{-2q}_{2(j-1)\alpha}(\lambda) \\
&= 4(q-(j-1))(\langle\alpha, i\lambda\rangle + 2q - 2(j-1))\,\Gamma^{-2q}_{2(j-1)\alpha}(\lambda).
\end{aligned}
$$

It suffices clearly to compute $\Gamma^{-2q}_{2j\alpha}(\lambda)$ for $j = 0, 1, \ldots, q$. All the other coefficients will be zero. Unfortunately, it is not so easy to prove the same result for $n > 2$.

We are, in this note, interested in the case $m = 4$ and consequently in the case $m = -2$.

Theorem 4.2 *Suppose $i\lambda(H) = \sum_{k=1}^{n} a_k\, H_k$ where $n = 2$, 3 or 4. Then, the coefficients in the series in (6) for $m = -2$, subject to the recursive relation in (7) and the condition $\Gamma_0^{-2}(\lambda) = \prod_{p<q}(1-(a_p-a_q)/2)$ are all polynomials. Furthermore, there are only a finite number of nonzero coefficients. They are given in Appendix A.*

The choice of $\Gamma_0^{-2}(\lambda)$ in the theorem forces all the coefficients to be polynomial functions. Later Theorem 4.3 will provide a further justification for that choice.

Proof:

$n = 2$: We have shown above that the series will terminate (2 nonzero terms).

$n = 3$: Formula (7) is easy to program on the computer; we have used Mathematica for these computations. We need to compute $g_{ij} = \Gamma^{-2}_{2(i\alpha+j\beta)}(\lambda)$. We have

$$
m_{ij}\, g_{ij} = \sum_{p<i} a_{pj}\, g_{pj} + \sum_{q<j} b_{iq}\, g_{iq} + \sum_{l<\min\{i,j\}} c_{i-l\,j-l}\, g_{i-l\,j-l}. \tag{8}
$$

The main point is to show that the series will terminate, that is, that all but a finite number of the g_{ij} are 0. We computed the terms g_{ij} for i and j between 0 and 10 and found that, for these indices, $g_{ij} = 0$ if $2 < i \leq 10$ or $2 < j \leq 10$.

Suppose $g_{ij} \neq 0$ for i or j greater than 2. We must have i or j greater than 10. Choose i and j having this property with $i + j$ minimal.

Based on the hypothesis above, we add or subtract terms that are 0 when we decrease i and/or j by 1 in the right hand side of (8) if they are greater than 3.

For example, if $i > 10$ and $j \leq 3$, then $\sum_{p<i} a_{pj}\, g_{pj} = \sum_{p<i-1} a_{pj}\, g_{pj}$, $\sum_{q<j} b_{iq}\, g_{iq} = 0 = \sum_{q<j} b_{i-1\,q}\, g_{i-1\,q}$ and $\sum_{l<\min\{i,j\}} c_{i-l\,j-l}\, g_{i-l\,j-l} = 0 = \sum_{l<\min\{i-1,j\}} c_{i-1-l\,j-l}\, g_{i-1-l\,j-l}$.

So in any case the right hand side of (8) is equal to $m_{i-1\,j}\, g_{i-1\,j}$, $m_{i\,j-1}\, g_{i\,j-1}$ or $m_{i-1\,j-1}\, g_{i-1\,j-1}$, all of which are 0.

$n = 4$: We need to compute $g_{ijk} = \Gamma^{-2}_{2(i\alpha+j\beta+k\gamma)}(\lambda)$. We have

$$\begin{aligned} m_{ijk}\, g_{ijk} &= \sum_{p<i} a_{pjk}\, g_{pjk} + \sum_{q<j} b_{iqk}\, g_{iqk} + \sum_{r<k} c_{ijr}\, g_{ijr} \\ &\quad + \sum_{l<\min\{i,j\}} d_{i-l\,j-l\,k}\, g_{i-l\,j-l\,k} + \sum_{l<\min\{j,k\}} e_{i\,j-l\,k-l}\, g_{i\,j-l\,k-l} \\ &\quad + \sum_{l<\min\{i,j,k\}} f_{i-l\,j-l\,k-l}\, g_{i-l\,j-l\,k-l}. \end{aligned} \tag{9}$$

We computed the terms g_{ijk} for i, j and k between 0 and 10. We found that, for these indices, $g_{ijk} = 0$ if $4 < i \leq 10$, $4 < j \leq 10$ or $4 < k \leq 10$.

Suppose $g_{ijk} \neq 0$ for i, j or k greater than 4. We must have i, j or k greater than 10. Choose i, j and k having this property with $i + j + k$ minimum.

It suffices in (9), to decrease i, j and/or k by one if they are greater than 5. The rest is as in the case $n = 3$. ∎

This proof shows that to prove the corresponding result for any given n, there is a finite amount of checking to do (as long as it is true !).

Theorem 4.3 *For $n = 2$, $n = 3$ and $n = 4$, let*

$$\mathbf{D} = e^{\rho/2} \sum_{\mu \in \Lambda} e^{-\mu}\, \Gamma_\mu^{-2} \left(\frac{\partial}{\partial H_1}, \ldots, \frac{\partial}{\partial H_n} \right) \tag{10}$$

(this is a finite sum and, since the Γ_μ^{-2} are polynomial functions, it makes sense to use differential operators with constant coefficients as arguments).

Then, $\mathcal{A}^{-1} = C\, \frac{1}{d^3}\, \mathbf{D} \circ \partial(\pi)$.

Proof: It is clear that $\frac{1}{d^3}\, \mathbf{D} \circ \partial(\pi)\, e^{i\lambda} = C\, \Phi_\lambda / c(-\lambda)$. ∎

The result for $n = 2$ is clearly not new. In the case $n = 3$, the operator $\mathbf{D}$ was given earlier in Theorem 3.1. R. J. Beerends obtained an operator corresponding to $\mathbf{D}$ for the case $n = 4$ but could only give it using another coordinate system, "the z-coordinates" (see [4]). Our formulation will permit us to prove Theorem 5.1 in Section 5.

5 Anker's conjecture

It is easy to show that for every symmetric space of noncompact type, we can write

$$P_t(e^H) = C\, e^{-\|\rho\|^2 t}\, t^{-q/2}\, e^{-r^2/(4t)}\, \phi_0(H)\, V_t(H)$$

where $\phi_0(H)$ is the Legendre function, $\lim_{t\to\infty} V_t(H) = 1$ and q, the dimension at infinity, is a known positive integer ($q = n^2$ in the case of $\mathbf{Pos}(n, \mathbf{H})$). The behaviour of the Legendre function is well known for symmetric spaces of noncompact type (see [1] by Jean-Philippe Anker).

In his paper [2], Jean-Philippe Anker conjectures that for every symmetric space of noncompact type, we have

$$V_t(H) \le C \prod_{\eta \in \Sigma_0^+} \left(1 + \frac{1+\eta}{t}\right)^{(m_\eta + m_{2\eta})/2-1} \tag{11}$$

for a positive constant C (Σ_0^+ is the set of indivisible positive roots). Note that Anker formulates his conjecture differently.

He proves his conjecture for the spaces $\mathbf{U}(p,q)/\mathbf{U}(p) \times \mathbf{U}(q)$ in [2] and points out that it is also true for all symmetric spaces of rank 1. The result is also is true for the complex case since $V_t(H) = 1$ for all H on such spaces (see [6]). We proved in [15] that for the space $\mathbf{Pos}(3, \mathbf{R})$, $V_t(H)$ is bounded above and below by constant multiples of the right hand side of (11). The corresponding results for the heat kernels of the real hyperbolic spaces have been obtained by E. B. Davies and N. Mandouvalos (see Theorem 5.7.2 in [5]).

Theorem 5.1 *Anker's conjecture, as given in (11), is valid for the spaces* $\mathbf{Pos}(n, \mathbf{H})$ *for* $n = 2$, *3 and 4.*

Proof: The result is already known for the case $n = 2$.

With the use of Theorem 4.3, the proof follows the same lines in all three cases. We will go over the case $n = 4$.

$P_t(e^H) = \mathcal{A}^{-1} \left[C\, e^{-\|\rho\|^2 t}\, t^{-4/2}\, e^{-r^2/(4t)}\right] = C\, d^{-3}(H)\, \mathbf{D}\, \partial(\pi) \left[e^{-\|\rho\|^2 t}\, t^{-4/2}\, e^{-r^2/(4t)}\right]$. As a consequence, we can set $\sum_{k=0}^{6} R_k(\alpha, \beta, \gamma)\, t^{-k} = e^{-\rho(H)/2}\, d^3(H)\, \phi_0(H)\, V_t(H)$.

We only have to show that there exist positive constants C_k $(0 \le k \le 6)$ such that

$$R_k(\alpha, \beta, \gamma)/(e^{-\rho(H)/2}\, d^3(H)\, \phi_0(H)) \le C_k \sum_{\substack{p_{ij} \in \{0,1\} \\ \sum_{i<j} p_{ij} = k}} \prod_{i<j} (1 + H_i - H_j)^{p_{ij}}. \tag{12}$$

Now, as shown in [1], there exists $C > 0$ such that

$$C^{-1} \prod_{i<j} (1 + H_i - H_j)\, e^{-\rho(H)} \le \phi_0(H) \le C \prod_{i<j} (1 + H_i - H_j)\, e^{-\rho(H)}.$$

This reduces the proof of equation (12) to

$$R_k(\alpha, \beta, \gamma)/d^3(H) \le C_k\, e^{-3\rho(H)/2} \sum_{\substack{p_{ij} \in \{0,1\} \\ \sum_{i<j} p_{ij} = k}} \prod_{i<j} (1 + H_i - H_j)^{1+p_{ij}}. \tag{13}$$

It actually suffices to prove that for some positive constants C_k $(0 \le k \le 6)$,

$$R_k(\alpha, \beta, \gamma) \le C_k \sum_{\substack{p_{ij} \in \{0,1\} \\ \sum_{i<j} p_{ij} = k}} \prod_{i<j} (H_i - H_j)^{1+p_{ij}} \tag{14}$$

for $H_i - H_j \ge M$ where $M > 0$ is any fixed positive number. Indeed, if all the roots are away from 0, $d^{-3}(H)$ behaves like $e^{-3\rho(H)/2}$. If exactly one of the roots, say α, is close to 0 we use the following device:

$$\begin{aligned} R_k(\alpha, \beta, \gamma)/d^3(H) &= \frac{1}{2} \frac{\alpha^3}{\sinh^3 \alpha} \frac{1}{(\sinh \beta \sinh(\alpha + \beta))^3} \int_0^1 (1-t)^2\, (R_k)_{111}(t\alpha, \beta, \gamma)\, dt \\ &\le C\, e^{-3\rho(H)/2} \int_0^1 |(R_k)_{111}(t\alpha, \beta, \gamma)|\, dt \end{aligned}$$

(the index 1 represents differentiation with respect to the first argument). We use here the fact that $R_k(\alpha, \beta, \gamma)/d^3(H)$ is analytic and, hence, the relevant derivatives are 0. Now,

$$\sum_{k=0}^{6} R_k(\alpha, \beta, \gamma)\, t^{-k} = t^6\, e^{r^2/(4t)} \sum_{\mu \in \Lambda} e^{-\mu}\, \Gamma_\mu^{-2} \left(\frac{\partial}{\partial H_1}, \ldots, \frac{\partial}{\partial H_n} \right) \partial(\pi)\, e^{-r^2/(4t)} \tag{15}$$

so the functions R_k belong to the algebra generated by the roots and by the exponential terms $e^{-\mu}$ $(\mu \in \Lambda)$. The inequality (14) above can only be improved by the differentiation and the integration above (modulo a multiplicative constant). The same reasoning applies if more than one root is close to 0.

We now have to prove (14). The only summands on the right hand side of (15) that have a significant contribution to the size of the functions $R_k(\alpha,\beta,\gamma)$ are those where $\mu = 2(i\alpha + j\beta + k\gamma)$ with $ijk = 0$; as for the other summands, the factor $e^{-\mu}$ eclipses any polynomial terms in the roots. If we refer to Appendix A, we only need to consider $t^6\, e^{r^2/(4t)}\, e^{-\mu}\, \Gamma_\mu^{-2}\left(\frac{\partial}{\partial H_1},\dots,\frac{\partial}{\partial H_n}\right) \partial(\pi)\, e^{-r^2/(4t)}$ where μ is one of of 0, 2α, 2β, 2γ, $4\beta+2\gamma$, $2\beta+4\gamma$, $4\beta+4\gamma$, $4\alpha+2\beta$, $2\alpha+2\gamma$, $2\alpha+4\beta$, $4\alpha+4\beta$, $2\beta+2\gamma$ or $2\alpha+2\beta$.

The first 11 terms are really like $t^6\, e^{r^2/(4t)}\, \Gamma_0^{-2}\left(\frac{\partial}{\partial H_1},\dots,\frac{\partial}{\partial H_n}\right) \partial(\pi)\, e^{-r^2/(4t)}$ (or smaller) and the last 2 are of the same type.

We collect the coefficients of t^{-k}, $0 \le k \le 6$. Away from the walls of the Weyl chamber, the right hand side of (14) is like $\alpha\,\beta^2\,\gamma^{3+k} + \alpha^2\,\beta\,\gamma^{3+k} + \alpha\,\beta^{4+k}\,\gamma + \alpha^{3+k}\,\beta^2\,\gamma + \alpha^{3+k}\,\beta\,\gamma^2$. We just have to make sure that the terms $\alpha^p\,\beta^q\,\gamma^r$ that make up the coefficients of t^{-k} are such that $p+q+r \le 6+k$, that $p+q$, $q+r$ and $p+r$ are less than or equal to $5+k$ and that $p \le 3+k$, $r \le 3+k$, $q \le 4+k$. The rest can be done by inspection (using a computer !). ∎

A Appendix

We write $\lambda(H) = \sum_{k=1}^n a_k H_k$. The identities seen below between the coefficients can easily be explained. The spherical functions can be given as

$$\phi_\lambda = \sum_{s\in W} c(s\lambda)\,\Phi_{s\lambda} = C\,\frac{1}{d^3}\,\pi(\lambda)\,|c(\lambda)|^2 \sum_{s\in W} (\det s)\; e^{i\,s\lambda+\rho/2}\,\Gamma_\mu^{-2}(s\lambda)\,e^{-\mu}$$

($\pi(\lambda) = \prod_{i<j}(a_i - a_j)$). The identities follow from the Weyl invariance of the spherical functions. This reasoning can be applied only if the series in (6) has finitely many terms (otherwise the series is convergent only in the positive Weyl chamber).

Let $\alpha = \alpha(H) = H_1 - H_2$, $\beta = \beta(H) = H_2 - H_3$ and $\gamma = \gamma(H) = H_3 - H_4$. We will write alternatively $\Gamma_\mu^{-2}(a_1, \dots, a_n)$ instead of $\Gamma_\mu^{-2}(\lambda)$. Recall that $\Gamma_0^{-2}(a_1, \dots, a_n) = \prod_{i<j}(1-(a_i-a_j)/2)$. We give below the other nonzero coefficients.

$n = 2$: $\Gamma_{2\alpha}^{-2}(\lambda) = \Gamma_0^{-2}(a_2, a_1)$.

$n = 3$: $\Gamma_{2\alpha}^{-2}(\lambda) = \Gamma_0^{-2}(a_2, a_1, a_3)$, $\Gamma_{2\beta}^{-2}(\lambda) = \Gamma_0^{-2}(a_1, a_3, a_2)$, $\Gamma_{2\alpha+4\beta}^{-2}(\lambda) = \Gamma_0^{-2}(a_3, a_1, a_2)$, $\Gamma_{4\alpha+2\beta}^{-2}(\lambda) = \Gamma_0^{-2}(a_2, a_3, a_1)$, $\Gamma_{4\alpha+4\beta}^{-2}(\lambda) = \Gamma_0^{-2}(a_3, a_2, a_1)$ and $\Gamma_{2\alpha+2\beta}^{-2}(\lambda) = 6 - \sum_{i<j}((a_i - a_j)/2)^2$.

$n = 4$: $\Gamma_{2\alpha}^{-2}(\lambda) = \Gamma_0^{-2}(a_2, a_1, a_3, a_4)$, $\Gamma_{2\beta}^{-2}(\lambda) = \Gamma_0^{-2}(a_1, a_3, a_2, a_4)$, $\Gamma_{2\gamma}^{-2}(\lambda) = \Gamma_0^{-2}(a_1, a_2, a_4, a_3)$,

$\Gamma_{4\beta+2\gamma}^{-2}(\lambda) = \Gamma_0^{-2}(a_1, a_3, a_4, a_2)$, $\Gamma_{2\beta+4\gamma}^{-2}(\lambda) = \Gamma_0^{-2}(a_1, a_4, a_2, a_3)$,

$\Gamma_{4\beta+4\gamma}^{-2}(\lambda) = \Gamma_0^{-2}(a_1, a_4, a_3, a_2)$, $\Gamma_{4\alpha+2\beta}^{-2}(\lambda) = \Gamma_0^{-2}(a_2, a_3, a_1, a_4)$,

$\Gamma_{2\alpha+2\gamma}^{-2}(\lambda) = \Gamma_0^{-2}(a_2, a_1, a_4, a_3)$, $\Gamma_{2\alpha+4\beta}^{-2}(\lambda) = \Gamma_0^{-2}(a_3, a_1, a_2, a_4)$,

$\Gamma_{4\alpha+4\beta}^{-2}(\lambda) = \Gamma_0^{-2}(a_3, a_2, a_1, a_4)$, $\Gamma_{6\alpha+4\beta+4\gamma}^{-2}(\lambda) = \Gamma_0^{-2}(a_2, a_4, a_3, a_1)$,

$\Gamma_{6\alpha+4\beta+2\gamma}^{-2}(\lambda) = \Gamma_0^{-2}(a_2, a_3, a_4, a_1)$, $\Gamma_{4\alpha+2\beta+4\gamma}^{-2}(\lambda) = \Gamma_0^{-2}(a_2, a_4, a_1, a_3)$,

$\Gamma_{2\alpha+6\beta+2\gamma}^{-2}(\lambda) = \Gamma_0^{-2}(a_3, a_1, a_4, a_2)$, $\Gamma_{6\alpha+6\beta+2\gamma}^{-2}(\lambda) = \Gamma_0^{-2}(a_3, a_2, a_4, a_1)$,

$\Gamma_{4\alpha+8\beta+4\gamma}^{-2}(\lambda) = \Gamma_0^{-2}(a_3, a_4, a_1, a_2)$, $\Gamma_{6\alpha+8\beta+4\gamma}^{-2}(\lambda) = \Gamma_0^{-2}(a_3, a_4, a_2, a_1)$,

$\Gamma_{2\alpha+4\beta+6\gamma}^{-2}(\lambda) = \Gamma_0^{-2}(a_4, a_1, a_2, a_3)$, $\Gamma_{2\alpha+6\beta+6\gamma}^{-2}(\lambda) = \Gamma_0^{-2}(a_4, a_1, a_3, a_2)$,

$\Gamma_{4\alpha+4\beta+6\gamma}^{-2}(\lambda) = \Gamma_0^{-2}(a_4, a_2, a_1, a_3)$, $\Gamma_{6\alpha+6\beta+6\gamma}^{-2}(\lambda) = \Gamma_0^{-2}(a_4, a_2, a_3, a_1)$,

$\Gamma_{4\alpha+8\beta+6\gamma}^{-2}(\lambda) = \Gamma_0^{-2}(a_4, a_3, a_1, a_2)$, $\Gamma_{6\alpha+8\beta+6\gamma}^{-2}(\lambda) = \Gamma_0^{-2}(a_4, a_3, a_2, a_1)$,

$\Gamma_{2\beta+2\gamma}^{-2}(\lambda) = (1 - \prod_{i=2}^{4}(a_1 - a_i)/2)\,(6 - \sum_{2\leq i<j\leq 4}((a_i - a_j)/2)^2)$,

$\Gamma_{2\alpha+2\beta}^{-2}(\lambda) = \Gamma_{2\beta+2\gamma}^{-2}(-a_4, -a_1, -a_2, -a_3)$, $\Gamma_{6\alpha+6\beta+4\gamma}^{-2}(\lambda) = \Gamma_{2\beta+2\gamma}^{-2}(-a_1, -a_2, -a_3, -a_4)$,

$\Gamma_{4\alpha+6\beta+6\gamma}^{-2}(\lambda) = \Gamma_{2\beta+2\gamma}^{-2}(a_4, a_1, a_2, a_3)$, $\Gamma_{2\alpha+6\beta+4\gamma}^{-2}(\lambda) = \Gamma_{2\beta+2\gamma}^{-2}(-a_2, -a_1, -a_3, -a_4)$,

$\Gamma_{4\alpha+6\beta+2\gamma}^{-2}(\lambda) = \Gamma_{2\beta+2\gamma}^{-2}(a_3, a_1, a_2, a_4)$, $\Gamma_{2\alpha+2\beta+4\gamma}^{-2}(\lambda) = \Gamma_{2\beta+2\gamma}^{-2}(-a_3, -a_1, -a_2, -a_4)$,

$$\begin{aligned}\Gamma^{-2}_{2\alpha+2\beta+2\gamma}(\lambda) &= 36 - 28\,[(a_1-a_3)/2+(a_2-a_4)/2] + 4\,[(a_1-a_3)/2+(a_2-a_4)/2]^2 \\ &\quad + [(a_1-a_3)/2+(a_2-a_4)/2]^3 - [(a_1-a_3)/2+(a_2-a_4)/2]^4/4 \\ &\quad + ((a_1-a_2)/2)^2\,((a_3-a_4)/2)^2\,(5/2 - 2\,[(a_1-a_3)/2+(a_2-a_4)/2]) \\ &\quad + (((a_1-a_2)/2)^2 + ((a_3-a_4)/2)^2)\,(-12 + 7\,[(a_1-a_3)/2+(a_2-a_4)/2]) \\ &\quad - (((a_1-a_2)/2)^2 + ((a_3-a_4)/2)^2)\,[(a_1-a_3)/2+(a_2-a_4)/2]^2/2 \\ &\quad + 3\,(((a_1-a_2)/2)^4 + ((a_3-a_4)/2)^4)/4,\end{aligned}$$

$\Gamma^{-2}_{2\alpha+4\beta+2\gamma}(\lambda) = \Gamma^{-2}_{2\alpha+2\beta+2\gamma}(a_1,a_3,a_2,a_4)$, $\Gamma^{-2}_{4\alpha+4\beta+2\gamma}(\lambda) = \Gamma^{-2}_{2\alpha+2\beta+2\gamma}(a_3,a_2,a_1,a_4)$,

$\Gamma^{-2}_{4\alpha+6\beta+4\gamma}(\lambda) = \Gamma^{-2}_{2\alpha+2\beta+2\gamma}(a_3,a_4,a_2,a_1)$, $\Gamma^{-2}_{4\alpha+4\beta+4\gamma}(\lambda) = \Gamma^{-2}_{2\alpha+2\beta+2\gamma}(a_2,a_4,a_1,a_3)$,

$\Gamma^{-2}_{2\alpha+4\beta+4\gamma}(\lambda) = \Gamma^{-2}_{2\alpha+2\beta+2\gamma}(a_1,a_4,a_3,a_2)$.

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