A global estimate for the Legendre function for the root systems of type A with arbitrary multiplicities *

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Abstract

We generalize the global estimate Jean-Philippe Anker found for the Legendre function ϕ_0 on symmetric spaces of noncompact type to the root systems of type A with arbitrary multiplicities.

1 Introduction and preliminaries

We refer the reader to [2, 3] for the general background information. For more specific background, the reader should also consult the paper [4].

If λ is a complex-valued linear functional on $\mathfrak{a}$, the spherical functions on the symmetric space of noncompact type G/K are given by

$$\phi_\lambda(g) = \int_K e^{(i\,\lambda-\rho)(\mathcal{H}(g\,k))}\,dk$$

where $g = k\,e^{\mathcal{H}(g)}\,n \in K\,A\,N$ (Iwasawa decomposition) and ρ is the half-sum of the roots.

In [1], Anker showed that for $H \in \overline{\mathfrak{a}^+}$,

$$\phi_0(e^H) \asymp e^{-\rho(H)} \prod_{\alpha>0} (1+\alpha(H)) \tag{1}$$

where $f(x) \asymp g(x)$ for $x \in D$ means that $f(x)/g(x)$ is bounded above and below on D by strictly positive constants. This result applies to every symmetric space of noncompact type.

Let $\mathfrak{a} = \{(H_1,\dots,H_r)\colon\ H_i \in \mathbf{R}\}$. The root system of type A_{r-1} is given by the set $R = \{e_i - e_j\colon\ i \neq j\}$ of linear functionals on $\mathfrak{a}$ where $e_i(H) = H_i$. The Weyl group is the set $W = S_r$ which acts on $\mathfrak{a}$ by permuting the indices and on a linear functional λ by $(w\cdot\lambda)(H) = \lambda(w^{-1}\cdot H)$. Note that, usually, we would consider the subset of $\mathfrak{a}$ orthogonal to R namely the set of H's with $\sum_{k=1}^r H_k = 0$ (refer to Remark 2 below). However, for our purposes, it is convenient to use the definition given above. We write $\mathfrak{a}^+ = \{H \in \mathfrak{a}\colon H_i - H_j > 0 \text{ whenever } i > j\}$.

In [7], we showed that the spherical functions on symmetric cones satisfy a recurrence relation (refer to (2) below) involving the multiplicity m of the roots. This allows us to give the following generalization of the zonal spherical functions for the root systems of type A.

Definition 1 *Let $H = (H_1,\dots,H_r) \in \mathfrak{a}^+$. Let $E(H) = \{\xi = (\xi_1,\dots,\xi_{r-1})\colon\ H_1 \geq \xi_1 \geq H_2 \geq \xi_2 \geq H_3 \geq \dots \geq H_{r-1} \geq \xi_{r-1} \geq H_r\}$. Let*

$$S_m(\xi,H) = d(H)^{1-m}\,d_1(\xi)^{1-m}\left[\pm\prod_{i=1}^{r-1}\prod_{j=1}^{r}\sinh(\xi_i - H_j)\right]^{m/2-1},$$

$$d(H) = \prod_{i<j}\sinh(H_i - H_j), \qquad d_1(\xi) = \prod_{i<j}\sinh(\xi_i - \xi_j),$$

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AMS Subject Classification: 33C52 (43A90, 53C35) *where the sign $\pm$ is chosen so that $\prod_{i=1}^{r-1}\prod_{j=1}^{r}\sinh(\xi_i - H_j) \geq 0$ on the set $E(H)$.*

The generalized spherical functions are given by

$$\phi_\lambda^{(m)}(e^H) = \frac{\Gamma(m\,r/2)}{(\Gamma(m/2))^r}\, e^{i\,a_r\,\sum_{k=1}^r H_k} \int_{E(H)} \phi_{\lambda_0}^{(m)}(e^\xi)\, S_m(H,\xi)\, d_1^m(\xi)\, d\xi \tag{2}$$

where $\lambda(H) = \sum_{j=1}^r a_j\, H_j$ and $\lambda_0(\xi) = \sum_{i=1}^{r-1}(a_i - a_r)\,\xi_i$. If $r = 1$, then $\phi_\lambda^{(m)}(e^H) = e^{i\,\lambda(H)}$.

Remark 2 *If λ is as in the definition and $\lambda^c(H) = \sum_{j=1}^r (a_j + c)\, H_j$ then*

$$\phi_{\lambda^c}^{(m)}(e^H) = e^{c\,\sum_{j=1}^r H_k}\, \phi_\lambda^{(m)}(e^H)$$

(straightforward if one uses induction).

Definition 1 corresponds to the spherical functions in the case of $\mathfrak{sl}(n,\mathbf{R})$ (with $m = 1$), in the case of $\mathfrak{sl}(n,\mathbf{C})$ (with $m = 2$), in the case of $\mathfrak{su}^*(2\,n)$ (with $m = 4$) and in the exceptional case $e_{6(-26)}$ (with $r = 3$ and $m = 8$).

The generalization of the generalized spherical functions given in Definition 1 is equivalent to the one found in [4] with some notational adjustments although their definition is not recursive. It is shown in [4] that these spherical functions satisfy a hypergeometric system of differential equations (refer to [6]) which corresponds, in the case of a symmetric space of noncompact type, to the fact that the spherical functions are joint eigenfunctions of the left-invariant differential operators on the space. This result could also be shown using induction and (2).

In [7], we used the same idea to extend the relationship between the spherical functions on the symmetric cones (multiplicity m, $m = 1$, 2 and 4) and the Jack polynomials of index $\alpha = 2/m$ to all values of m thus giving a recursive formula for the Jack polynomials.

2 Global estimate

We will show that Anker's estimate (1) remains valid for the root systems A_{r-1} with arbitrary positive multiplicities.

Theorem 3 *Suppose $m > 0$. Then*

$$\phi_0^{(m)}(e^H) \asymp e^{-\rho^{(m)}(H)} \prod_{i<j}(1 + H_i - H_j)$$

where $\rho^{(m)}(H) = \frac{m}{2}\sum_{i<j}(H_i - H_j)$.

Proof: We use induction on n. The result is clearly true for $r = 1$.

Assume that the result is valid for $r - 1$. Using the induction hypothesis and the fact that for $x \geq 0$, $\sinh x \asymp x\, e^x/(1+x)$, we have

$$\begin{aligned}\phi_0^{(m)}(e^H) &\asymp e^{-\frac{m}{2}\sum_{i<j}(H_i - H_j)} \frac{\prod_{i<j}(H_i - H_j)^{1-m}}{\prod_{i<j}(1 + H_i - H_j)^{1-m}} \int_{E(H)} \prod_{i<j}(\xi_i - \xi_j) \\ &\qquad \cdot \left(\prod_{j=1}^r \left[\prod_{i=1}^{j-1} \frac{\xi_i - H_j}{1 + \xi_i - H_j} \prod_{i=j}^{r-1} \frac{H_j - \xi_i}{1 + H_j - \xi_i}\right]\right)^{m/2-1} d\xi \\ &= C_m \frac{e^{-\rho^{(m)}(H)}}{\prod_{i<j}(1 + H_i - H_j)^{1-m}} \int_\sigma (K(\xi,H))^{1-m/2}\, (\beta_1 \cdots \beta_r)^{m/2-1}\, d\beta\end{aligned}$$

where $\beta_k = \frac{\prod_{i=1}^{r-1}(\xi_i - H_k)}{\prod_{i\neq k}(H_i - H_k)}$, $1 \le k \le r$, $\sigma = \{(\beta,\ldots,\beta_r)\colon \beta_i \ge 0, \sum_{i=1}^r \beta_i = 1\}$ and $K(\xi,H) = \prod_{j=1}^r \left[\prod_{i=1}^{j-1}(1+\xi_i - H_j)\prod_{i=j}^{r-1}(1+H_j-\xi_i)\right]$.

The theorem is a consequence of the following inequalities: if $\xi \in E(H)$ then

$$\beta_1\cdots\beta_r \prod_{i<j}(1+H_i-H_j)^2 \le K(\xi,H) \le \prod_{i<j}(1+H_i-H_j)^2.$$

Indeed,

$$\begin{aligned}
&\beta_1\cdots\beta_r \prod_{i<j}(1+H_i-H_j)^2\\
&= \prod_{j=1}^r \left[\prod_{i=1}^{j-1}\frac{(\xi_i-H_j)\,(1+H_i-H_j)}{H_i-H_j}\prod_{i=j}^{r}\frac{(H_j-\xi_i)\,(1+H_j-H_{i+1})}{H_j-H_{i+1}}\right]\\
&\le \overbrace{\prod_{j=1}^r \left[\prod_{i=1}^{j-1}(1+\xi_i-H_j)\prod_{i=j}^{r}(1+H_j-\xi_i)\right]}^{K(\xi,H)} \qquad (3)\\
&\le \prod_{j=1}^r \left[\prod_{i=1}^{j-1}(1+H_i-H_j)\prod_{i=j}^{r}(1+H_j-H_i)\right] = \prod_{i<j}(1+H_i-H_j)^2.
\end{aligned}$$

To obtain (3), we used the fact that $\xi \in E(H)$, we have $(\xi_i - H_j)\,(1+H_i-H_j) \le (H_i - H_j)\,(1+\xi_i - H_j)$ if $i<j$ and $(H_j-\xi_i)\,(1+H_j-H_{i+1}) \le (H_j - H_{i+1})\,(1+H_j-\xi_i)$ if $j \le i$. ∎

The theorem partially extends to complex values of m in the following manner.

Corollary 4 *Suppose $\Re m > 0$ ($\Re m$ is the real part of m). Then there exists a constant $C_m > 0$ such that*

$$|\phi_0^{(m)}(e^H)| \le C_m\, e^{-\rho^{(\Re m)}(H)} \prod_{i<j}(1+H_i-H_j).$$

Proof: Using induction, it is straightforward to show that $|\phi^{(m)}(e^H)| \le C_m\,\phi^{(\Re m)}(e^H)$ for some positive constant C_m. ∎

Theorem 3 and its corollary can be used to give an estimate for all spherical functions.

We first explain the situation in the context of symmetric spaces of noncompact type. Definition 5 and Proposition 6 are given in that context.

Definition 5 *Let $\mathfrak{a}^*$ be the space of real-valued linear functionals on $\mathfrak{a}$. We define $\mathfrak{a}^*_+ = \{\lambda \in \mathfrak{a}^*\colon\ \langle\lambda,\alpha\rangle > 0\ \forall\ \alpha > 0\}$. If $\lambda \in \mathfrak{a}^*$, there exists a unique $w \in W$ such that $w\cdot\lambda \in \overline{\mathfrak{a}^*_+}$. We will then write $\lambda_+ = w\cdot\lambda$ for this choice of w.*

The following result holds for all symmetric spaces:

Proposition 6 *Suppose $H \in \overline{\mathfrak{a}^+}$. Then*

$$|\phi_\lambda(e^H)| \le e^{\Re(i\,\lambda)^+(H)}\,\phi_0(e^H).$$

Proof: We know that $\phi_\lambda^{(m)} = \phi_{w\cdot\lambda}^{(m)}$ whenever $w \in W$ (refer to [3, Theorem 4.3, Chapter IV]). We can therefore assume that $\Re(i\,\lambda) = \Re(i\,\lambda)^+$. Let $C(H) = H(e^H\,K)$ which is the convex hull of the set $W \cdot H$. Recalling that the maximum of a linear function on a convex set will take place at one of its vertices, we have

$$\begin{aligned}|\phi_\lambda(e^H)| &\le \int_K e^{\Re(i\,\lambda)(H(e^H\,k))}\, e^{-\rho(H(e^H\,k))}\,dk \\ &\le \max_{F\in C(H)} e^{\Re(i\,\lambda)(F)} \int_K e^{-\rho(H(e^H\,k))}\,dk = \max_{w\in W} e^{\Re(i\,\lambda)(w\cdot H)}\,\phi_0(e^H).\end{aligned}$$

Given [2, Theorem 2.22, Chapter 7], the result follows. ∎

We now show that a similar result holds for the root systems of type A with generalized multiplicities.

If $\gamma(H) = \sum_{k=1}^r \gamma_k\, H_k$ with γ_k real then we write $\gamma^+(H) = \sum_{k=1}^r \gamma_{w(k)}\, H_k$ where $w \in W$ is chosen so that the sequence $\gamma_{w(k)}$ is decreasing. If we normalized $\mathfrak{a}$ by taking $\sum_{k=1}^r H_k = 0$ then this would corresponds exactly to Definition 5.

Corollary 7 *Suppose $\Re m > 0$. Then there exists a constant $D_m > 0$ such that*

$$|\phi_\lambda^{(m)}(H)| \le D_m\, e^{\Re(i\,\lambda)^+(H)}\, e^{-\rho^{(\Re m)}(H)} \prod_{i<j}(1 + H_i - H_j).$$

Proof: We know that $\phi_\lambda^{(m)} = \phi_{w\cdot\lambda}^{(m)}$ whenever $w \in W$ (refer to [5, Theorem 2.2]). We can therefore assume that $\Re(i\,\lambda) = \Re(i\,\lambda)^+$. This implies that $\Re(i\,\lambda_0) = \Re(i\,\lambda_0)^+$ in (2) and we can use induction (the result is clearly true for $r = 1$):

$$\begin{aligned}|\phi_\lambda^{(m)}(e^H)| &\le C_m\, e^{\Re(i\,a_r)\sum_{k=1}^r H_k} \int_{E(H)} |\phi_{\lambda_0}^{(m)}(e^\xi)|\, S_{\Re m}(H,\xi)\, d_1^{\Re m}(\xi)\,d\xi \\ &\le C_m'\, e^{\Re(i\,a_r)\sum_{k=1}^r H_k} \int_{E(H)} e^{\Re(i\,\lambda_0)(\xi)}\, e^{-\rho^{(m)}(\xi)} \prod_{i<j}(1+\xi_i-\xi_j)\, S_{\Re m}(H,\xi)\, d_1^{\Re m}(\xi)\,d\xi.\end{aligned}$$

The result follows as in the proof of the theorem if we note that

$$\begin{aligned}e^{\Re(i\,a_r)\sum_{k=1}^r H_k}\, e^{\Re(i\,\lambda_0)(\xi)} &= e^{\sum_{k=1}^{r-1}\Re(i\,(a_k-a_r))\,(\xi_k-H_k)} + e^{\sum_{k=1}^r \Re(i\,a_k)\,H_k} \\ &\le e^{\sum_{k=1}^r \Re(i\,a_k)\,H_k} = e^{\Re(i\,\lambda)(H)}\end{aligned}$$

since $\xi_k - H_k \le 0$ and $\Re(i\,(a_k - a_r)) \ge 0$ for each $k < r$. ∎

3 Conclusion

Theorem 3 and the results that follow illustrate the advantages of having a recursive definition of the generalized spherical functions.

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