A global estimate for the Legendre function for the root systems of type $A$ with arbitrary multiplicities

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Abstract

We generalize the global estimate Jean-Philippe Anker found for the Legendre function $\phi_0$ on symmetric spaces of noncompact type to the root systems of type $A$ with arbitrary multiplicities.

1 Introduction and preliminaries

We refer the reader to [2, 3] for the general background information. For more specific background, the reader should also consult the paper [4].

If $\lambda$ is a complex-valued linear functional on $a$, the spherical functions on the symmetric space of noncompact type $G/K$ are given by

$$\phi_\lambda(g) = \int_K e^{(i\lambda - \rho)(H(gk))} \, dk$$

where $g = k e^{H(g)} n \in K A N$ (Iwasawa decomposition) and $\rho$ is the half-sum of the roots.

In [1], Anker showed that for $H \in a^+$,

$$\phi_0(e^H) \approx e^{-\rho(H)} \prod_{\alpha > 0} (1 + \alpha(H))$$  \hspace{1cm} (1)

where $f(x) \asymp g(x)$ for $x \in D$ means that $f(x)/g(x)$ is bounded above and below on $D$ by strictly positive constants. This result applies to every symmetric space of noncompact type.

Let $a = \{(H_1, \ldots, H_r): H_i \in \mathbb{R}\}$. The root system of type $A_{r-1}$ is given by the set $R = \{e_i - e_j: i \neq j\}$ of linear functionals on $a$ where $e_i(H) = H_i$. The Weyl group is the set $W = S_r$ which acts on $a$ by permuting the indices and on a linear functional $\lambda$ by $(w \cdot \lambda)(H) = \lambda(w^{-1} \cdot H)$.

Note that, usually, we would consider the subset of $a$ orthogonal to $R$ namely the set of $H$’s with $\sum_{k=1}^r H_k = 0$ (refer to Remark 2 below). However, for our purposes, it is convenient to use the definition given above. We write $a^+ = \{H \in a: H_1 - H_j > 0\}$ whenever $i > j$.

In [7], we showed that the spherical functions on symmetric cones satisfy a recurrence relation (refer to (2) below) involving the multiplicity $m$ of the roots. This allows us to give the following generalization of the zonal spherical functions for the root systems of type $A$.

**Definition 1** Let $H = (H_1, \ldots, H_r) \in a^+$. Let $E(H) = \{\xi = (\xi_1, \ldots, \xi_{r-1}): H_1 \geq \xi_1 \geq H_2 \geq \xi_2 \geq H_3 \geq \cdots \geq H_{r-1} \geq \xi_{r-1} \geq H_r\}$. Let

$$S_m(\xi, H) = d(H)^{1-m} d_1(\xi)^{1-m} \left[ \pm \prod_{i=1}^{r-1} \prod_{j=1}^r \sinh(\xi_i - H_j) \right]^{m/2-1},$$

$$d(H) = \prod_{i < j} \sinh(H_i - H_j), \quad d_1(\xi) = \prod_{i < j} \sinh(\xi_i - \xi_j),$$

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where the sign ± is chosen so that \( \prod_{i=1}^{r-1} \prod_{j=1}^{r} \sinh \left( \xi_i - H_j \right) \geq 0 \) on the set \( E(H) \).

The generalized spherical functions are given by

\[
\phi_\lambda^{(m)}(e^H) = \frac{\Gamma(m r/2)}{(\Gamma(m/2))^r} e^{i \alpha} \sum_{k=1}^{r} H_k \int_{E(H)} \phi_{\lambda_0}^{(m)}(e^\xi) S_m(H, \xi) d^m(\xi) d\xi
\]

(2)

where \( \lambda(H) = \sum_{j=1}^{r} a_j H_j \) and \( \lambda_0(\xi) = \sum_{i=1}^{r-1} (a_i - a_r) \xi_i \). If \( r = 1 \), then \( \phi_\lambda^{(m)}(e^H) = e^{i \lambda(H)} \).

Remark 2 If \( \lambda \) is as in the definition and \( \lambda^c(H) = \sum_{j=1}^{r} (a_j + c) H_j \) then

\[
\phi_{\lambda^c}^{(m)}(e^H) = e^c \sum_{k=1}^{r} H_k \phi_\lambda^{(m)}(e^H)
\]

(straightforward if one uses induction).

Definition 1 corresponds to the spherical functions in the case of \( \mathfrak{s}(n, \mathbb{R}) \) (with \( m = 1 \)), in the case of \( \mathfrak{s}(n, \mathbb{C}) \) (with \( m = 2 \)), in the case of \( \mathfrak{su}^*(2n) \) (with \( m = 4 \)) and in the exceptional case \( \mathfrak{e}_{6(-26)} \) (with \( r = 3 \) and \( m = 8 \)).

The generalization of the generalized spherical functions given in Definition 1 is equivalent to the one found in [4] with some notational adjustments although their definition is not recursive. It is shown in [4] that these spherical functions satisfy a hypergeometric system of differential equations (refer to [6]) which corresponds, in the case of a symmetric space of noncompact type, to the fact that the spherical functions are joint eigenfunctions of the left-invariant differential operators on the space. This result could also be shown using induction and (2).

In [7], we used the same idea to extend the relationship between the spherical functions on the symmetric cones (multiplicity \( m, m = 1, 2 \) and \( 4 \)) and the Jack polynomials of index \( \alpha = 2/m \) to all values of \( m \) thus giving a recursive formula for the Jack polynomials.

### 2 Global estimate

We will show that Anker’s estimate (1) remains valid for the root systems \( A_{r-1} \) with arbitrary positive multiplicities.

**Theorem 3** Suppose \( m > 0 \). Then

\[
\phi_0^{(m)}(e^H) \asymp e^{-\rho^{(m)}(H)} \prod_{i<j} (1 + H_i - H_j)
\]

where \( \rho^{(m)}(H) = \frac{m}{2} \sum_{i<j} (H_i - H_j) \).

**Proof:** We use induction on \( n \). The result is clearly true for \( r = 1 \).

Assume that the result is valid for \( r - 1 \). Using the induction hypothesis and the fact that for \( x \geq 0, \sinh x \asymp x e^x/(1 + x) \), we have

\[
\phi_0^{(m)}(e^H) \asymp e^{-\frac{m}{2} \sum_{i<j} (H_i - H_j)} \prod_{i<j} (1 + H_i - H_j)^{1-m} \int_{E(H)} \prod_{i<j} (\xi_i - \xi_j)
\]

\[
\cdot \left( \prod_{j=1}^{r} \left[ \prod_{i=1}^{j-1} \frac{\xi_i - H_j}{1 + \xi_i - H_j} \prod_{i=j}^{r-1} \frac{H_j - \xi_i}{1 + H_j - \xi_i} \right] \right)^{m/2-1} d\xi
\]

\[
= C_m \prod_{i<j} (1 + H_i - H_j)^{1-m} \int_{\sigma} (K(\xi, H))^{1-m/2} (\beta_1 \cdots \beta_r)^{m/2-1} d\beta
\]
where $\beta_k = \prod_{j=k}^{r-1} (\xi_j - H_k), \ 1 \leq k \leq r, \ \sigma = \{ (\beta, \ldots, \beta_r) : \beta_i \geq 0, \sum_{i=1}^{r} \beta_i = 1 \}$ and $K(\xi, H) = \prod_{j=1}^{r} \left[ \prod_{i=1}^{j-1} (1 + \xi_i - H_j) \prod_{i=j}^{r-1} (1 + H_j - \xi_i) \right].$

The theorem is a consequence of the following inequalities: if $\xi \in E(H)$ then

$$\beta_1 \cdots \beta_r \prod_{i<j} (1 + H_i - H_j)^2 \leq K(\xi, H) \leq \prod_{i<j} (1 + H_i - H_j)^2.$$

Indeed,

$$\beta_1 \cdots \beta_r \prod_{i<j} (1 + H_i - H_j)^2
= \prod_{j=1}^{r} \left[ \prod_{i=1}^{j-1} \frac{(\xi_i - H_j)(1 + H_i - H_j)}{H_i - H_j} \prod_{i=j}^{r} \frac{(H_j - \xi_i)(1 + H_j - H_{i+1})}{H_j - H_{i+1}} \right]
\leq \prod_{j=1}^{r} \left[ \prod_{i=1}^{j-1} (1 + \xi_i - H_j) \prod_{i=j}^{r} (1 + H_j - \xi_i) \right]
\leq \prod_{j=1}^{r} \left[ \prod_{i=1}^{j-1} (1 + H_i - H_j) \prod_{i=j}^{r} (1 + H_j - H_i) \right] = \prod_{i<j} (1 + H_i - H_j)^2. \quad (3)$$

To obtain (3), we used the fact that $\xi \in E(H)$, we have $(\xi_i - H_j)(1 + H_i - H_j) \leq (H_i - H_j)(1 + \xi_i - H_j)$ if $i < j$ and $(H_j - \xi_i)(1 + H_j - H_{i+1}) \leq (H_j - H_{i+1})(1 + H_j - \xi_i)$ if $j \leq i$.

The theorem partially extends to complex values of $m$ in the following manner.

**Corollary 4** Suppose $\mathfrak{R}m > 0$ ($\mathfrak{R}m$ is the real part of $m$). Then there exists a constant $C_m > 0$ such that

$$|\phi_0^{(m)}(e^H)| \leq C_m e^{-\rho(\mathfrak{R}m)(H)} \prod_{i<j} (1 + H_i - H_j).$$

**Proof:** Using induction, it is straightforward to show that $|\phi^{(m)}(e^H)| \leq C_m \phi^{(\mathfrak{R}m)}(e^H)$ for some positive constant $C_m$. □

Theorem 3 and its corollary can be used to give an estimate for all spherical functions.

We first explain the situation in the context of symmetric spaces of noncompact type. Definition 5 and Proposition 6 are given in that context.

**Definition 5** Let $\mathfrak{a}^*$ be the space of real-valued linear functionals on $\mathfrak{a}$. We define $\mathfrak{a}^*_+ = \{ \lambda \in \mathfrak{a}^* : \langle \lambda, \alpha \rangle > 0 \ \forall \ \alpha > 0 \}$. If $\lambda \in \mathfrak{a}^*$, there exists a unique $w \in W$ such that $w \cdot \lambda \in \mathfrak{a}^*_+$. We will then write $\lambda_+ = w \cdot \lambda$ for this choice of $w$.

The following result holds for all symmetric spaces:

**Proposition 6** Suppose $H \in \mathfrak{a}^*_+$. Then

$$|\phi_\lambda(e^H)| \leq e^{\mathfrak{R}(i\lambda)^+(H)} \phi_0(e^H).$$
Proof: We know that \( \phi^{(m)}_\lambda = \phi^{(m)}_{w,\lambda} \) whenever \( w \in W \) (refer to [3, Theorem 4.3, Chapter IV]). We can therefore assume that \( \Re(i \lambda) = \Re(i \lambda)^+ \). Let \( C(H) = H(e^H K) \) which is the convex hull of the set \( W \cdot H \). Recalling that the maximum of a linear function on a convex set will take place at one of its vertices, we have

\[
|\phi^{(m)}_\lambda(e^H)| \leq \int_K e^{\Re(i \lambda)(H(e^H k))} e^{-\rho(H(e^H k))} \, dk \\
\leq \max_{F \in C(H)} e^{\Re(i \lambda)(F)} \int_K e^{-\rho(H(e^H k))} \, dk = \max_{w \in W} e^{\Re(i \lambda)(w^H)} \phi_0(e^H).
\]

Given [2, Theorem 2.22, Chapter 7], the result follows. \( \blacksquare \)

We now show that a similar result holds for the root systems of type \( A \) with generalized multiplicities.

If \( \gamma(H) = \sum_{k=1}^r \gamma_k H_k \) with \( \gamma_k \) real then we write \( \gamma^+(H) = \sum_{k=1}^r \gamma_w(k) H_k \) where \( w \in W \) is chosen so that the sequence \( \gamma_w(k) \) is decreasing. If we normalized \( a \) by taking \( \sum_{k=1}^r H_k = 0 \) then this would corresponds exactly to Definition 5.

Corollary 7 Suppose \( \Re m > 0 \). Then there exists a constant \( D_m > 0 \) such that

\[
|\phi^{(m)}_\lambda(H)| \leq D_m e^{\Re(i \lambda)^+(H)} e^{-\rho^{(m)}(H)} \prod_{i<j} (1 + H_i - H_j).
\]

Proof: We know that \( \phi^{(m)}_\lambda = \phi^{(m)}_{w,\lambda} \) whenever \( w \in W \) (refer to [5, Theorem 2.2]). We can therefore assume that \( \Re(i \lambda) = \Re(i \lambda)^+ \). This implies that \( \Re(i \lambda_0) = \Re(i \lambda_0)^+ \) in (2) and we can use induction (the result is clearly true for \( r = 1 \):

\[
|\phi^{(m)}_\lambda(e^H)| \leq C_m e^{\Re(i \lambda_0) \sum_{k=1}^r H_k} \int_{E(H)} |\phi^{(m)}_\lambda(e^\xi)| S_{\Re m}(H, \xi) d_{\Re m}^{(m)}(\xi) d\xi \\
\leq C_m e^{\Re(i \lambda_0)(\xi)} e^{-\rho^{(m)}(\xi)} \prod_{i<j} (1 + \xi_i - \xi_j) S_{\Re m}(H, \xi) d_{\Re m}^{(m)}(\xi) d\xi.
\]

The result follows as in the proof of the theorem if we note that

\[
e^{\Re(i a_r) \sum_{k=1}^r H_k} e^{\Re(i \lambda_0)(\xi)} = e^{\sum_{k=1}^{r-1} \Re(i (a_k-a_r)) (\xi_k - H_k)} + e^{\sum_{k=1}^r \Re(i a_k) H_k} \leq e^{\sum_{k=1}^r \Re(i a_k) H_k} = e^{\Re(i \lambda)(H)}
\]

since \( \xi_k - H_k \leq 0 \) and \( \Re(i (a_k-a_r)) \geq 0 \) for each \( k < r \). \( \blacksquare \)

3 Conclusion

Theorem 3 and the results that follow illustrate the advantages of having a recursive definition of the generalized spherical functions.
References


