Singing sands, booming dune sands, and the stick-slip effect

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Abstract: The origin of the acoustic and seismic emissions from impacted singing grains and from avalanching dune sand grains is sought in modes of vibration in discreet grain columns. It is postulated that when the grains in a column are pressed together and forced to slide over one another, elastic shear bands are formed at the contact areas with distinct elastic moduli. Such contact shear bands would have implications in the formulation of the Hertz-Mindlin contact theory. The assembly of all grain columns below the impacting pestle forms the slip (slide) shear band. The transfer of energy from the pestle to the modes of vibration in such columns is effected by the stick-slip effect. The intense collective vibration of all columns in the slip shear band results in the familiar musical sound. The concept of grain flowability is used to justify the disparity between the acoustic emissions from impacted singing grains and from avalanching booming dune sand grains. The concept of grain columns is assumed to apply in the freely avalanching sand band, but with longer length to justify the lower frequencies. This approach predicts frequency spectra comprising a low frequency content and a dominant frequency with its harmonics in agreement with the experimental evidence. Additionally, it can account for the low frequency vibration evoked when booming sand flows through a funnel, with implications in the understanding of grain silo vibrations. It is argued that sand grains do not sing or boom since the stick-slip effect in not applicable in the contact shear bands.

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1. **Introduction**

The mechanism responsible for the seismic and acoustic emissions, when a bed of singing sand or silica gel grains is impacted by a pestle, was the subject of a recent paper by Patitsas [1]. The mechanism was sought in shear modes of vibration in a well defined slip channel (shear band) comprising several grain layers ahead of the impacting pestle. It was argued that due to some not yet understood physico-chemical effect on the grain surface, a bed of singing (musical) grains is characterized by a relatively high level of rigidity. Thus, the grains just ahead of the pestle are subjected to a relatively high stress level resulting in the partial fluidization (softening) of the tips of the grain asperities and any grain coating at the contact areas, resulting in turn in reduced elastic moduli in the slip channel. Apart from the contact shear bands, the grains are viewed as rigid bodies slipping (sliding) over one another.

However, the assumption in [1] that the length of the slip channel, in directions normal to the direction of the pestle motion, is also well defined needs reconsideration. Such an assumption was necessary in order to view the slip channel as a thin cavity with well defined walls, with well defined shear standing wave patterns and with well defined eigenfrequencies. It is highly likely that the walls at the channel ends are not well defined and its length is not very large compared to its thickness. Whereas, it could be argued that the boundary of the slip channel adjacent to the pestle is well defined, it is difficult to argue that this is also the case at the lower boundary. A more gradual transition from relatively low column stiffness below the pestle to nominal values in the grain bed, well below the pestle, is a more realistic assumption. Furthermore, it is unreasonable to expect that continuum mechanics alone can provide a satisfactory solution when the width of the channel is only about ten times the particle (grain) size. Thus, the determination of the frequencies of the seismic and acoustic emissions remains an open question.

Similarly, there are serious questions that need be considered regarding the recent approaches in accounting for the seismic and acoustic emissions from avalanching booming dune sands. Four such approaches have been published over the past eight years: In the mainly experimental report by Andreotti [2], it is shown that during a booming dune avalanche, there are elastic waves propagating along the dune surface extending several cm below the surface. It is then argued that the grains would oscillate according to the particle displacement dictated by such waves. Furthermore, such waves would synchronize the grain-grain collisions and would become excited by such collisions. It is known that when grains of any kind are induced to avalanche down an inclined plane, the average time required by one grain to overtake another is given by the expression, $T_c = (1/0.4)\sqrt{\bar{d}/g}$, where $\bar{d}$ is the average grain diameter and $g = 9.8m/s^2$, Andreotti [2], MiDi [3]. Thus, the dominant frequency of the propagating waves and that of the seismic and acoustic emissions can be defined as $f_d = 1/T_c$. However, in the paper by Vriend at al. [4], it is reported that dune vibrations were detected even when there was no apparent avalanche in progress and moreover, during avalanches the dominant frequency f_d was accompanied by several harmonics.

Subsequent reports by Bonneau et al. [5, 6] go to great lengths to elucidate the properties of waves propagating along a dune surface where the elastic moduli increase with sand depth. However, there is no clear identification of the modes corresponding to frequencies equal to multiples of f_d, and in the latest report by Andreotti and Bonneau [7], the question of harmonics of f_d is not addressed. In this latter report, it is assumed that a thin shear band is formed between the avalanching sand band and the static sand below, and that leads to the excitation of the surface waves. Whereas, such a shear band is nearly evident when a sand plate breaks off and begins to slide downhill, there is no such evidence after the plate breaks up and a free avalanche ensues. On page 253 in Bagnold [8], it is stated that the velocity of a grain layer at depth ζ decreases linearly with depth until it is zero at some depth $\zeta = h_o$.

Furthermore, by studying a video recording by The National Geographic Society, Survivors of the Skeleton Coast Park, Namibia, Africa (1993), it can be concluded that the avalanche front looses height gradually until it comes to rest when the height is about 2 or 3 cm. By observing the tail-end of the avalanches in YouTube presentations, it can be inferred that its thickness is considerably lower than in the front. One such presentation by the authors of the paper by Douady et al. [9], with the title: The Song of the Dunes, can be reached from the link in the same paper. Additionally, when the gate on a wood frame on the slip face of a boomable dune is suddenly released, with sand height behind the gate up to 10 cm, [9], it is difficult to imagine that the thickness of the avalanching band is the same in its entire length. Therefore, the argument in [5] that for booming to occur the avalanche thickness must exceed a certain threshold is rather tenuous. The argument is more tenuous when applied to the case of a sand pile pushed by a blade, [9], where the geometry is even more ill-defined.

When, about 0.5 kg of booming sand grains from Sand Mountain, Nevada, USA, were placed in a glass jar, 7 cm in diameter by 16 cm in length, and shaken horizontally along the jar axis, the dominant frequency of the acoustic emission was 280 Hz, Leach and Rubin [10]. It can be argued that there is grain layer rollover and a high stress level when the grain mass collides with the jar wall, as there is layer rollover and high stress level when the avalanche front collides with the static sand ahead. The dominant frequency, f_d, is about four times larger in the former than in the latter case. According to Nori et al. [11], f_d is approximately 65 Hz in the latter case. According to Leach et al. [12], f_d tends to decrease with increased grain mass in the jar. It appears that the mechanism responsible for the emissions from highly localized events could also be responsible for the emissions from large scale events such as the pushing of the sand mass by the hand or a blade and the sliding of sand plates on a dune surface. Furthermore, the booming (hum) sound emitted during a dune sand avalanche, where the grains are gravity driven, was observed in the laboratory when booming sand was poured from a hopper into a bag, Lewis [13].

In the report by Patitsas [1], the concept of a shear band (slip channel) several mm thick, under a sliding sand plate or under a freely avalanching sand band, was used to explain the relevant emissions as originating with shear modes of vibration in the channel with shear phase velocity about 1 m/s, such that $\lambda \approx$ twice the channel thickness. But, even if such a channel existed in the case of free avalanche, it would not be well defined at the lateral ends, as in the case of the slip channel under an impacting pestle. However, this approach could explain the harmonics of f_d.

In the experimental report in [9], it is also recognized that the frequency, f_d, is defined by the overtake time, T_c, but the synchronization of the collisions is effected by some sort of coupling between adjacent grain layers due to some wave that propagates up-down between the static sand and the surface of the avalanching band. There is no attempt to account for overtone frequencies, but such an approach would lead to overtones in the sequence of $3f_d$, $5f_d$ etc. However, the notion of up-down motion of grain layers is in agreement with the notion of up-down grain column oscillations proposed in this study.

In the mainly experimental report by Vriend et al. [4], the booming emission is sought in compression wave propagation along a surficial grain layer about 2 m in depth. The frequency is defined by the condition that for specific phase velocities in the substrate, the grain layer and in the air, there is total reflection at the boundaries. This approach was criticized by

Andreotti et al. [14] especially regarding the assumption that the propagation velocity does not increase with depth, and that there is experimental evidence that appreciable vibrations during booming are limited to the depth of only about 10 cm, [2]. No explanation is provided as to the mechanism that would allow for the conversion of the gravitational energy of the grains into elastic energy of the propagating modes in the surficial layer. Furthermore, the mathematical model that is invoked could not account for the low frequency content present in the free avalanche acoustic and seismic emissions, and then, it is highly unlikely that the energy generated by a few kg of freely avalanching sand grains into a hole dug on the face of a dune would be sufficient to excite a wave in such a large layer in thickness and length. The absence of boomability in certain dunes in a given area is not a strong indicator that the booming mechanism has to lie well beneath the dune surface, since on the surface all dunes appear to be the same. Equivalently, only certain sections of the Eastern and Northern shores of Lake Michigan USA, visited on August 2009, exhibited singability.

No explanation has been presented as to why booming dune sand would cease booming, when freely avalanching, but would continue to be musical when squeezed, Criswell et al. [15]. Furthermore, no arguments have been published as to the unexpected low propagation velocity of the synchronization wave on the booming dune surface [2], and more generally as to the unusually low propagation velocity of an elastic wave in a pile of ordinary sand, about 50 m/s, well below the values predicted by the Hertz-Mindlin theory, Bachrach et al. [16]. Finally, no satisfactory explanation exists as to why singing sands do not boom and booming sands do not sing, and as to why no acoustic emission is produced during an ordinary (silent) grain avalanche.

Before proceeding with the presentation of the present approach, it is deemed appropriate to attempt to elucidate the terminologies used in describing the sounds emitted by sheared granular media

. (a) Singing or squeaking sound refers to musical sound of short duration, up to 200 ms, with frequencies in the range 250 to 2500 Hz, emitted when beach sands or silica gel grains are impacted by an object or stepped on.

(b) Booming dune sound refers to a relatively low frequency, 60 to 100 Hz, continuous hum-like (drone-like) sound emitted when dune sand grains avalanche freely downhill.

(c) Continuous singing or squeaking sound refers to musical sound emitted when the bed of singing (squeaking) grains is sheared continuously by a vertical rod, immersed about 3 cm, or by a smooth object pulled along the grain surface. The known frequency range is under 1000 Hz.

(d) Pushed (squeezed) booming sand sound refers to a continuous musical sound emitted, roughly in the frequency range, 25 to 350 Hz, when the sand mass is pushed continuously by the hand or a blade on the face of a dune, or by a blade in a confined geometry.

(e) Roar sound refers to loud sound emitted when the booming sand is pushed downhill in a heaped-up manner. The frequency range has not been recorded but it is likely at about 200 Hz.

(f). Sand plate sound refers to the sound emitted when dune sand plates avalanche freely downhill. The frequency range has not been recorded but it is likely at about 200 Hz.

List of symbols used extensively

all parameters are expressed in the MKS system of units unless otherwise specified

d overall diameter of a grain

R R=d/2

$\bar{d}$ average diameter of grains in a column

b thickness of a contact shear band between two grains

R_b radius of a contact shear band

c_p compression phase velocity in a contact shear band

c_s shear phase velocity in a contact shear band

$\hat{x}$ unit vector along the direction of grain-grain slide

$\hat{z}$ unit vector along a grain column

$\hat{r}$ unit vector in the radial direction in a contact shear band

z distance from the bottom of a grain column towards the pestle above

θ polar angle around $\hat{z}$

α wavenumber along $\hat{z}$ equal to $2\pi/\lambda = \omega/c_s$

λ wavelength along $\hat{z}$

ζ sand depth in a dune face or in any sand bed

ξ_r particle displacement along $\hat{r}$ inside a contact shear band

ξ_z particle displacement along $\hat{z}$ inside a contact shear band

N number of grains in a column

V_c compression propagation velocity in a grain column

V_p compression propagation velocity in a grain bed

M_p equivalent pestle mass above a grain column

f_1 frequency of the fundamental mode in a grain column

$f_2, ..f_N$ overtone frequencies, on average, harmonics of f_1

f_p frequency of the overburden mode of vibration where the entire grain column acts like a sort spring

T_c time for one grain to overtake another$=(1/0.4)\sqrt{(\bar{d}/9.8)}$

f_c grain-grain collision frequency in a surface grain avalanche$=1/T_c$

f_d frequency at the center of the dominant experimental spectrum envelope

H height of drop of a pestle on the grain bed surface

Brevort River sand: singing sand collected at the mouth of the Brevort River flowing into the North shore of Lake Michigan, USA, about 25 km west from the city of St. Ignace.

Large plastic container: a 46×28 cm by 10 cm in sand depth plastic container

2. The grain column approach

Figure 1 depicts an assumed grain configuration inside a slip channel where the five grain layers slide over one another along $\hat{x}$. For reasons to become clear later, the slip channel can also be referred to as, **the vibration shear band** or

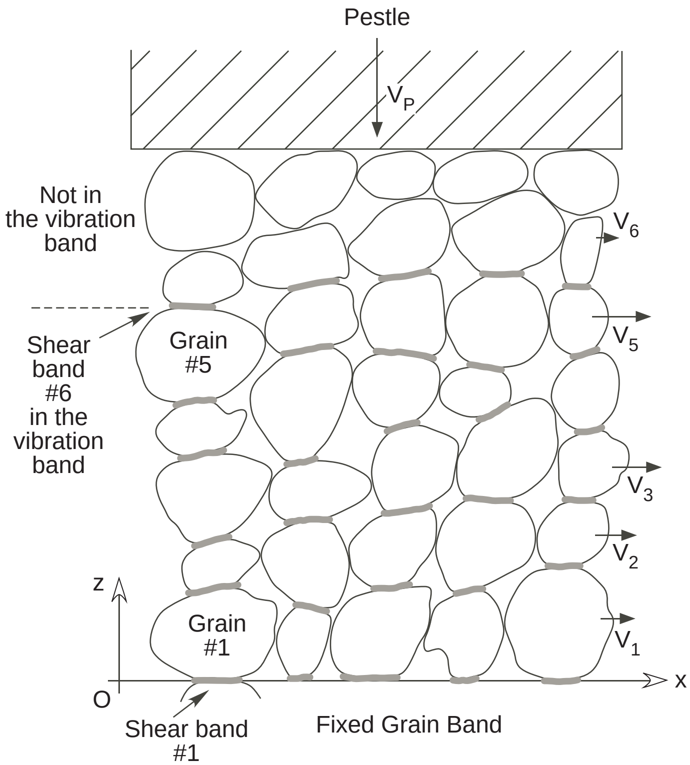
the column shear band or even better, **the slip (slide) shear band**. Ultimately the source of all vibrations are the elastic shear bands at the grain contact areas. For the first column on the left hand side, they are labeled as: shear band # 1 at the bottom to shear band #6 at the top. It is understood that the lifetime of a given column is roughly equal to the average time required for a grain to overtake another, T_o, and that the lifetime of the five column configuration is about five times shorter and that the lifetimes would decrease with increased grain number, N, in the columns. However, the duration of the signals generated when a small steel sphere impacts a grain bed, Fig. 2 for example, is comparable to the time $T_o \approx$35 ms for relative slippage velocity, 1 cm/s, and grain diameter, d=0.35 mm. As the sphere descends into the grain bed, a given column, about 5 to 10 mm long, looses grains at the top and gains grains at the bottom. There is also grain exchange between columns, but, the collective vibration of the columns, outlined below, would tend to smooth out the frequency shifts in the course of time.

These disk-like shear bands, due primarily to a physico-chemical change during a grain-grain slippage, are characterized by thickness, b, radius, R_b, and by compression and shear moduli that result in the corresponding phase velocities, c_p, c_s. The parameters, b and c_p, c_s are viewed as inherent properties of such bands, fairly independent of the confining pressure along the grain columns, while this would not be so for R_b. From (3) in [17], it is possible to determine the value of R_b when two ideally spherical and homogeneous grains, of radius R, are pressed together by the force, F, namely, $R_b = (C_1/C_2)^{1/3}$, where, $C_1 = 3FR(1-\nu)$ and $C_2 = 8G$, where, G, ν are the shear modulus and the Poisson's ratio of the grains. Thus, the contact area, S, scales as $R^{2/3}$.

The grains are not perfect spheres and the contact shear bands may occur at the asperity-asperity contacts, where R becomes R_c and $R_c << R$, where R_c is the asperity radius. If $R_c = R/27$, then, R_b and S are decreased by the factor of 9 and 3 respectively. However, in such a case, the single grain-grain contact area is replaced by several asperity-asperity contact areas, and if, on average, by three such contacts, then the total contact area, S, and the total grain-grain stiffness remain the same. The equations in Appendix A do not include the asperity radii, only the total contact areas, S_1, S_2 etc. As such, in what follows, grain-grain contacts will be considered only.

There is no information available that would lead to an estimate of the band thickness b. But, if R_c/b=20 and R_c=1750 nm, then b=87 nm and $\bar{d}/b$=4000 for $\bar{d}$=0.35 mm. It can be seen in (9) in appendix B and in the computations described below that f_d depends weakly on b, *i.e.*, as $b^{-0.5}$, so, the above estimate of $\bar{d}/b$ is not as critical as that of R_b. An estimate of R_b can be obtained from (3) in [17] with F equal to the gravitational force exerted on a grain when an 11 mm steel sphere is dropped on the Brevort River sand, the equivalent steel sphere mass on a grain column being equal to 162×10^{-7} kg. To this end, the sum of the cross-sectional areas of the columns below the sphere was assumed to be equal to 1/3 the sphere cross-sectional area. This results in $R_b = 1.3 \times 10^{-3}$mm and in $\bar{d}/R_b$=270 with $\bar{d}$=0.35 mm.

Fig.1. An assumed grain column configuration in a slip shear band. The shaded areas correspond to the elastic contact shear bands with physical properties of their own when the grains are forced to slide past one another. They are characterized by compression and shear phase velocities, c_p, c_s and particle displacements ξ_z, ξ_x.



In a recent report by Patitsas [18], it is demonstrated that the water layer on the epidermis of a finger rubbed on a glass surface acts as the interfacial band that facilitates slipping and also results in the decrease of the friction coefficient with relative velocity resulting in the stick-slip effect. Furthermore, it is argued that the shear modes of vibration responsible for the acoustic emission are to be found in the finger skin. However, there is no reason why the modes of vibration in the skin with thickness $b_s \approx$5 mm, shear phase velocity $c_s \approx$10 m/s and wavelength $\lambda \approx 2b_s$ could not also exist in the interfacial band with thickness $b \ll \lambda$. In this sense, the interfacial band becomes also the source of the energy of the modes of vibration in the skin band.

In the present case, the water interfacial band is replaced by the N contact shear bands and the skin band by the grain column. This can best be visualized by assuming that there is slippage only at the contact shear band # 1. The particle displacement oscillation in these contact bands, ξ_x, results in the conversion of the pestle to column vibration energy, while one grain slides over another, and this distinguishes primarily this approach from that in [1]. Such contact shear bands and the associated grain columns exist only during the pestle impact. Thus, detection of such bands by the use of elastic wave propagation through the grain mass would be rather difficult.

If the motion of the grains in a given column were along $\hat{z}$ only, the contact shear bands could be replaced by equivalent short weightless springs. It is a straightforward exercise to compute the eigenfrequencies and describe the corresponding modes of vibration for such a system. For N blocks and $N + 1$ springs, there are N modes of vibration with frequencies, $f_1, f_2 ... f_N$. For the mode with frequency f_1, all blocks oscillate in phase while for the mode with frequency f_N, neighboring blocks oscillate out of phase. The frequency f_1 tends to be rather insensitive to permutations of the blocks with different mass.

The stiffness of the grain contact shear bands is proportional to the contact area and to the square root of the elastic moduli of such bands, *i.e.*, proportional to the radius, R_b, and the compression and shear phase velocities, c_p, c_s, in such bands, as is demonstrated in (9) in appendix B. The frequency spectrum of a column vibration includes the frequency, f_1, that corresponds to the fundamental mode with $\lambda \approx 2N\bar{d} \approx$ twice the column length, the frequency, f_2 with $\lambda \approx N\bar{d}$, etc and the pestle frequency, f_p, that corresponds to the mode with $\lambda \gg N\bar{d}$, where the entire column acts like a short spring, Section 4.

It appears suitable at this stage to include a short paragraph from the study by Haff [19]. While the author was thinking of the booming dune emissions, the implications for the impacted grains are obvious. "Perhaps the mechanical analogue which most readily comes to mind is the slipstick phenomenon, a nonlinear mechanism by which a steady input of external energy is ultimately released and stored. This is certainly consistent with the oscillatory nature of the system and with its sensitivity to grain surface conditions and hence, presumably, to friction. To ascribe booming to a slipstick mechanism, however, is only to say the words; until we have a clear picture in mind of what the grains are actually doing, we do not really understand the origin of the booming sands".

3. Computations and implications

The question arises naturally as to the value of the number, N, of grains in a given grain column. Figures 8 and 9 in [1] depict the acoustic emissions when sand grains in a porcelain coffee cup, 9 cm in diameter, were impacted by a wood rod 25 mm in diameter. The sand depth before impact was about 10 mm and that after impact about 4 or 5 mm, implying that $N \approx 15$, however, in a regular grain bed several cm in depth, N=24 is a more realistic number. Then, with $\bar{d}$=0.35 mm, $\bar{d}/b$=4000, $\bar{d}/R_b$=270, estimated above, the following eigenfrequencies were computed, as outlined in Appendix A, f_p=67 Hz, f_1= 791 Hz, f_2= 1603 Hz, f_3=2561 Hz , etc by assuming that $c_p = c_s$=30 m/s.

The value, $\bar{d}$=0.35 mm corresponds to that of the singing sand grains collected from the mouth of the Brevort River flowing into the north shore of Lake Michigan, USA, about 25 km west from the city of St. Ignace, and the value, $\bar{d}/R_b$=270, is viewed as an average value along the grain column. The grain diameter, d_j, j in the range 1 to 24, was varied randomly between 0.2 and 0.5 mm and the circular contact areas, S_j in (7) in Appendix A, were evaluated by assuming the corresponding radii to be equal to the average diameter of the adjacent grains divided by 270. It may be noted that if $\bar{d}/b$ were equal to 1000, than, c_p would be equal to 60 m/s.

For all modes corresponding to these frequencies, the condition, b/λ <<1 is satisfied. The lowest frequency, f_p=67 Hz, corresponds to the pestle vibration and it decreases rather strongly with increased pestle mass M_p while the frequencies f_j decrease very weakly with increased M_p. This is in agreement with the lack of significant sensitivity of the frequency, $f_d = f_1$, on the length of the rod used to impact the grain bed or the manner by which the impaction is effected, by pushing or tapping the rod, for example.

Variations in the values of c_p, c_s revealed that the frequencies f_i are proportional to the values of c_p and nearly independent of the values of c_s. Only when c_s was reduced to $c_p/200$ there was an appreciable reduction in f_i. It could be argued that when $c_s = c_p$, $\alpha = \omega/c_s$ is extremely small, resulting in extremely large wavelength λ along the grain column compared with the contact band thickness, b. The lack of dependence of f_i on c_s can also be justified in the way the equations in Appendix A were based on the particle displacement along $\hat{z}$ as opposed to $\hat{x}$. The proportionality between the frequencies f_i and c_p implies that (9) in Appendix B represents adequately the compressive wave propagation in the grain column, since a given f_i, corresponding to a given standing wave pattern in the column, is proportional to the column propagation velocity, V_c, in (9) in Appendix B. The same argument can be repeated in justifying why the frequencies, f_1, f_2,.. are reduced by the factor of $\sqrt{2}$ when the ratio $\bar{d}/b$ is reduced by the factor of 2.0, why the same frequencies are increased by the factor 2 when the contact band radii are increased by the same factor, and why the same frequencies are reduced by the factor of 2.0 when the grain diameters are increased by the same factor.

The description of the corresponding modes was effected by computing the coefficients, $B_1, A_2, B_2, ..A_{13}, B_{13}$, Appendix A, and then by computing the particle (grain) displacements, ξ_z, ξ_r, at the bottom middle and top of every contact band, Appendix A. It was verified that, regardless of the number N of grains in the column, the particle displacement, ξ_z, behaves like a half sine function, *i.e.*, it peaks at the middle of the column and then dips to nearly zero at the top of the column . Then, according to (1, 2), in appendix A, ξ_r has a node at the middle of the column. Generally, ξ_r is about ten times as large as ξ_z. The relatively large particle displacement along $\hat{x}$ is consistent with the assumption that the stick-slip effect plays an important role towards the realization of the musical acoustic emission. In the case of the compression waves in the contact bands, $\xi_z > \xi_r$ and since ξ_z is constrained to remain small, such waves cannot become excited. This question is also raised in [18].

When the number of grains in a given column was decreased from 24 to 12, with the same average diameter, $\bar{d}$=0.35 mm, and the same value for c_p, the eigenfrequencies were increased by the factor of 2.0, since the propagation velocity in the column, V_c, remained the same but the wavelength was decreased by the same factor. Furthermore, when the grains were permuted in several ways, f_1 remained in the range, 800±80 Hz, provided the smaller grains were not appreciably segregated from the larger grains. The spread about the central value of f_1 lies within the half width of the major frequency envelopes, Fig. 2, for example. A spread of about ± 10 per cent was also determined around the frequency f_2.

The only known attempt to measure the propagation velocity in a grain bed of 28×28 cm by about 10 cm in depth can be found in the study by Liu and Nagel [20]. However, the grains were spherical glass beads 5 mm in diameter, *i.e.*, about 15 times larger than the sand grains used in this study. Additionally, the larger contact area and the thinner contact shear bands would result in larger stiffness and larger propagation velocity. The propagation velocity, by measuring the time of flight of a pulse, was about 280 m/s, but the group velocity, obtained from the change, with frequency, in the phase of the wave arriving at the detector was about 60 m/s. It is reported in [4] that "body wave velocities" in the range 180 to 300 m/s were measured well below the sand dune surface, and in Fig. 24.23 in Winterkorn and Fang [21], shear wave velocities as low as 150 m/s are reported for dry round and angular-grained sands. Additionally, wave velocities as low as 50 m/s in ordinary sand piles are reported in [16]. Evidently, the wave propagation velocity depends strongly on the depth of the propagation path between the points of emission (shot) and reception of the elastic wave. Then, there is the condition of a surface coating that would vary from place to place and from time to time. Additionally, it is reported in [16] that in preparing a sample from an unconsolidated material, in order to study the velocity of sound for example, its physical properties are disturbed.

4. The single spring model

In the study by Nishiyama and Mori [22], the authors attempt to explain the acoustic emissions from an impacted sand bed in terms of a single short spring action below the pestle. Towards this end, rods and disks of various diameters and lengths were dropped on the sand bed. There is an attempt to show that the dominant frequency, f_d, decreases with the rod mass, M, as $M^{-0.5}$. In particular brass rods (disks) with diameter D=2.5 cm and weight as low as 10 g were used. For a rod with M=100 g, the mass M_p on top of a grain column, $\bar{d}$=0.35 mm, amounts to 200×10^{-7} kg, while, in this study, in the case of the 11 mm steel sphere dropped on the grain bed, $M_p = 162 \times 10^{-7}$ kg. The grain column mass is, $M_c = 16.0 \times 10^{-7}$ kg for N=24.

For the small disk with M=20 g, the disk thickness amounts to only 4.8 mm. Thus, it is difficult to ascertain at what angle the small disks impacted the sand surface and the measurements with $M < 50$ g may not be considered. When rods with M in the range 50 to 600 g were dropped on a singing sand in a 40×40 cm by 30 cm deep container, f_d remained nearly constant at about 600 Hz, Fig. 2 in [22]. When 13 mm aluminum cylinders, ranging in length from 2 to 12 cm, were dropped vertically from a height of about 10 cm on the Brevort River sand in the 46×28 cm by 10 cm deep container, there were no significant changes in f_d. However, when singing sand grains and also silent glass beads, 0.6 mm in diameter, were placed in a 10 cm diameter porcelain mortar, but with only 1.0 cm in grain depth, f_d decreased as $M^{-0.5}$, Fig. 6 in [22]. It can be seen in [1] that in such confined geometry several kinds of grains in the same size range become singable to some extent.

The single spring model can be incorporated into the present approach by identifying the dominant frequency, f_d, with the pestle frequency, f_p, instead of with the fundamental, f_1. Thus, with N=24, $d_j = \bar{d}$=0.35 mm, c_p= 255 m/s, the eigenfrequencies were, f_p=815 Hz, f_1=10115 Hz, f_2=20173 Hz etc. Thus, there is only one frequency envelope centered at f_p, since the frequencies, f_1, f_2 are far beyond the range where the stick-slip effect would be applicable. The frequency f_p decreases exactly as $M^{-0.5}$, until it becomes practically zero, while f_1, f_2 hardly decrease. Such a value of c_p=255 m/s appears to be in contradiction with reported experimental values discussed in the previous section. The model would be a possible alternative if there were no harmonics of f_d, however, this is not the case as will be outlined in the next section.

In the context of the present approach, the decrease of f_d with cylinder mass, M, when the sand was impacted in a mortar with sand depth about 1 cm, could be attributed to grain slippage at the bottom of the mortar. For M roughly larger that 50 g, it could be assumed that the bottom of the grain columns is in contact with the mortar floor, where there would be appreciable slippage and reduced value of R_b, resulting in reduced f_d with increased M.

5. Experimental results and implications

The structure of the frequency spectrum of the signal emitted when a bed of singing grains is impacted by a pestle, or a freely falling object, depends on such parameters as: shape, size, surface texture, stiffness and speed of the impacting object, wall effects if the impactor is close to the walls of the container, and on the history of the grain bed, *i.e.* exposure to humidity and to previous impacts. Thus, it could be anticipated that a pencil-like sharp ended pestle would give rise to multiple frequencies corresponding to multiple slip shear bands around the sharp end. The scope of this study is not to determine the effect of any such parameters on the spectrum structure and in particular on the dominant frequency, f_d, but to develop a realistic theoretical model that would be based on the grain-grain interaction and would account for the spectral properties of isolated events. It would also account for changes of the spectral properties with one of the parameters listed above, when such changes have been reported in the literature. Thus, in the previous section, the model is used to explain the significant decrease of f_d with the mass of the impacting object when it was only a few mm away from the container wall when the sound was emitted. Furthermore, in the case of moist singing sand emitting sound with, $f_d \approx$2500 Hz, Brown et al. [23], it could be argued that the grain columns were very short and the sand mass relatively very stiff due to water content that would result in larger effective contact areas. Similar attempts will be made below regarding the sound from pushed (squeezed) booming dune sand, or the sound accompanying the flow of the same sand through a funnel. In this sense, the average value of f_d over several repetitions of the same event is not an an objective of this paper.

In the early stages of this study, Fall 2010, the sand was placed in a ceramic flower pot with 20 cm rim diameter and 10 cm depth. A regular pin microphone was placed about 10 cm roughly above and to the side from the point of impact, while the geophone was placed near the edge of the bed that was about 8 cm deep. Following an impact, the sand was poured into another container and then back into the flower pot, which was tapped lightly for better grain density uniformity. More recently, Fall 2011, the sand was placed in a larger plastic container, 46×28 cm by 10 cm in sand depth, with a regular thick towel placed inside the container before the sand was poured so as to prevent any slippage between the sand and the container walls. In such a large container, it was possible to effect several impacts, about 10 cm apart, before the sand was moved about in the container, with a small plastic bowl, and finally the surface was leveled and the container tapped lightly. The Geo Space Corporation geophone is omni-directional with natural frequency equal to 14 Hz suited for detecting relatively low frequency vibrations. The signals were processed by the NI USB-6210 analogue to digital converter and analyzed by the Lab-View Signal Express software of National Instruments.

5.1 Impacted singing sand grains

The purpose of this subsection is to demonstrate that harmonics of the fundamental can become excited when the grain bed is impacted by small steel spheres and by a glass rod, and thus, the single spring model is not applicable in these cases. Furthermore it is shown that the frequency plots are fairly well reproducible even over long time periods.

Fig.2. Frequency spectrum and the microphone recorded signal when an 11 mm steel sphere was dropped, height $H \approx$10 cm, on a Brevort River singing sand bed in a ceramic flower pot, 20 cm rim diameter by 10 cm in depth. $f_d \approx$790 Hz.

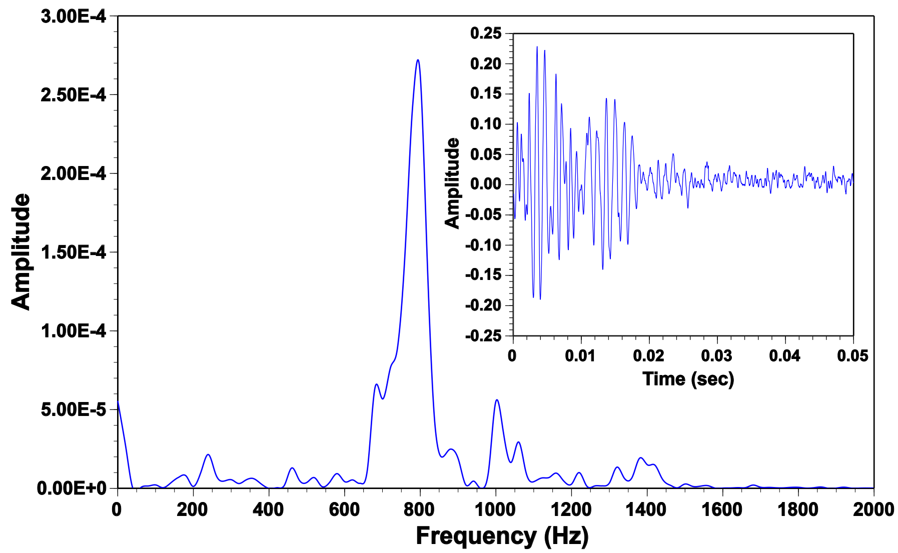
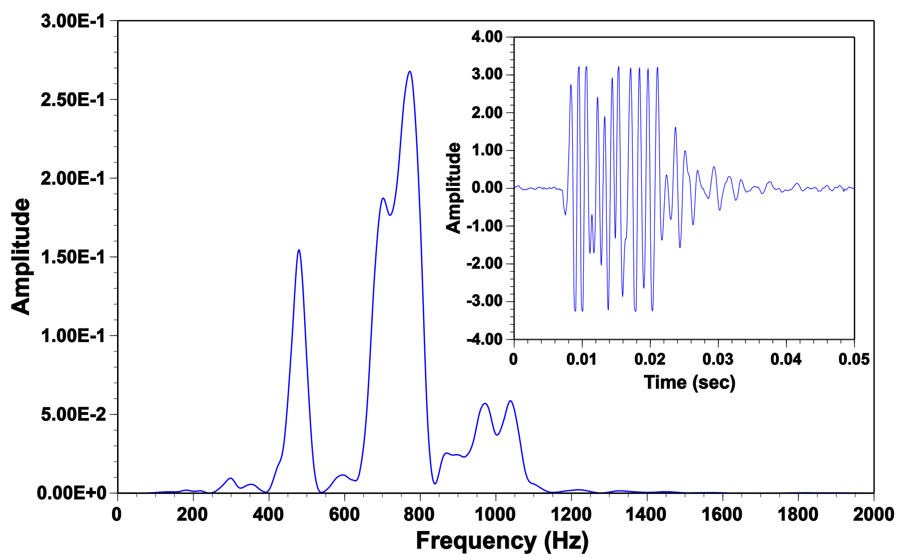


Fig.3. Same as in Fig. 2 but for the geophone signal. The geophone was placed near the pot rim about half immersed in the sand. $f_d \approx$770 Hz.



The same recordings were repeated recently, about one year later, with the same sand in the large plastic container. The microphone plots were reproduced quite well overall, however, that was not the case for the geophone plots, where, the slight elbow at about 695 Hz would appear as a totally separate peak. It could be argued that the geophone signal from the relatively small sphere is more susceptible to localized grain mass anisotropies as the sphere was well immersed in the sand after impact. The peak at about 475 Hz could be an indication of such an effect. It is likely due to the increase of the period of the inset signal after about 22 ms, implying that near the end of the impact, there was an appreciable increase in the grain column length in the slip shear band.

From the simple Fourier transform theory, the half width of the transform of the inset signal with length, Δt=18 ms in Fig. 2, is 56 Hz, and the separation of the side peaks from the center of the main envelope is 1.5×56=84 Hz. The half width of the main envelope is about 62 Hz, and the separation of the first side peak (elbow) on the right from the center, at 877 Hz, is 87 Hz. However, the separation of the first elbow on the left at 628 Hz amounts to 162 Hz, implying that it is not a Fourier side peak, and since it is also present in Fig.3, it could not be argued that it is due to some sort of noise, but rather due to a separate slip shear band.

The question of noise is raised below in connection with Fig. 15. Here, it could be defined as follows: It arises from the incoherent superposition of the elastic wave-trains emitted as a grain rubs its way past other grains, and of the wave-trains due to vibrations in grain columns of random length and direction around the pestle. Additionally, there would be low frequency content due to air mass accelerations recorded by the microphone. In this sense, the envelope at around 1000 Hz, in both plots, could be attributed to random column vibrations around the sphere, or due to such vibrations in a relatively thin and ill-defined slip side band. In this sense, when the slip shear band becomes totally ill-defined, then it becomes a source of noise. Thus, there could be two or more slip shear bands around the lower hemisphere of the impacting sphere, and around the pestle in general. A slight rotation of the sphere during the impact, or a non-vertical direction of the pestle, would be a cause of such multiplicity of slip shear bands. The lack of harmonics in the above plots could be attributed to the relatively short duration of the event and the lack of sufficient energy for the excitation of the corresponding modes.

Fig.4. Frequency spectrum of the microphone recorded signal when a 16 mm steel sphere was dropped on the Brevort River singing sand in the large plastic container, 46 $\times$ 28 cm by 10 cm sand depth, from the height $H \approx$20 cm. The vertical range was reduced somewhat in order for the first harmonic of $f_d \approx$578 Hz to be seen at about 1175 Hz. The second could be seen with further reduction at about 1780 Hz.

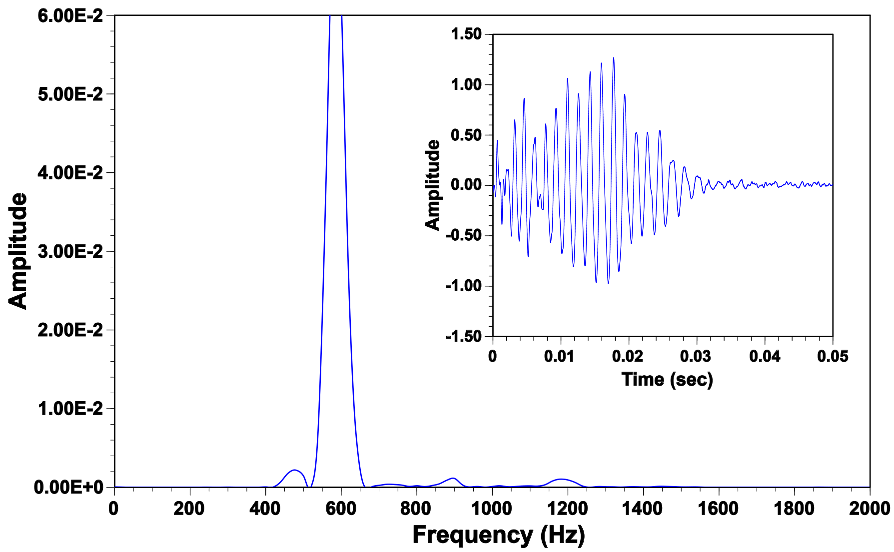


Fig.5. The same as in Fig. 4 but for the geophone recorded signal. f_d is estimated at about 590 Hz. The second harmonic at just under 1800 Hz is discernible.

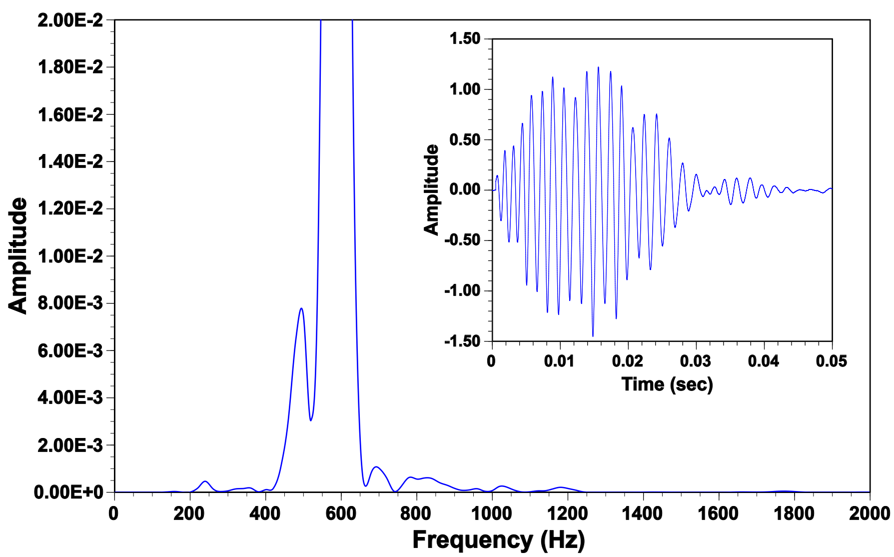


Fig.6. Same as in Fig. 4 but for a 25 mm steel sphere. f_d ≈467 Hz and the second harmonic is barely seen at about 1400 Hz.

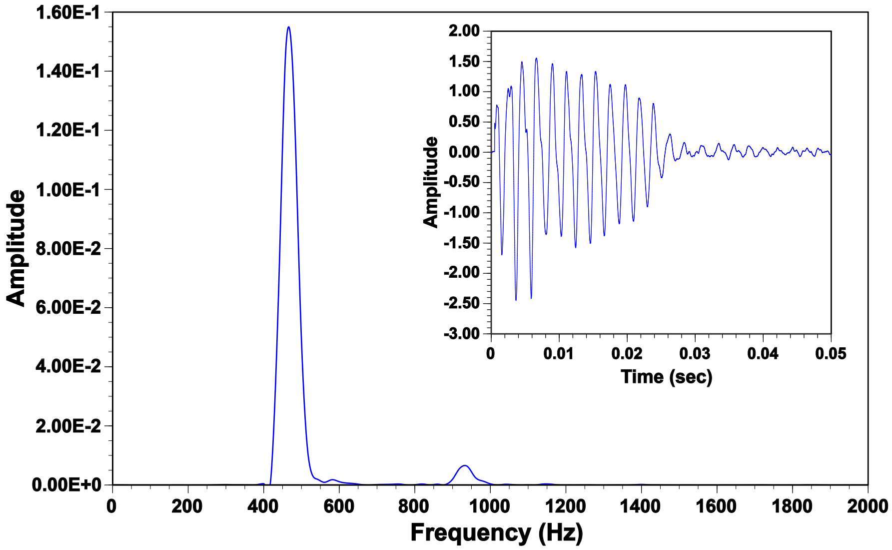


Fig.7. same as in Fig. 6 but for the geophone signal. The first harmonic of $f_d \approx 467$ Hz is quite prominent and the second is visible. The decrease of f_d from 590 Hz in Fig. 5 to 467 Hz in Fig. 7 could be attributed to the increase in the column number N.

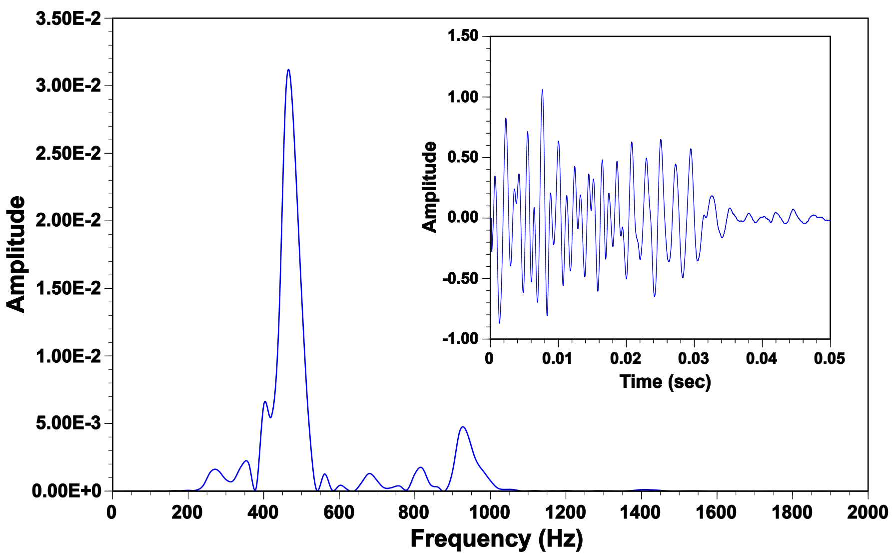


Fig.**8**. Frequency spectrum of the microphone recorded signal when a glass rod, length 7.5 cm, diameter 1.5 cm obtained from the Museum of Sand in Nima, Japan, was pushed manually into the Brevort River singing sand bed in the large plastic container, $f_d \approx$630 Hz. The side peak at 700 Hz is a Fourier side peak. The harmonics at about 1250 and 1870 Hz are discernible.

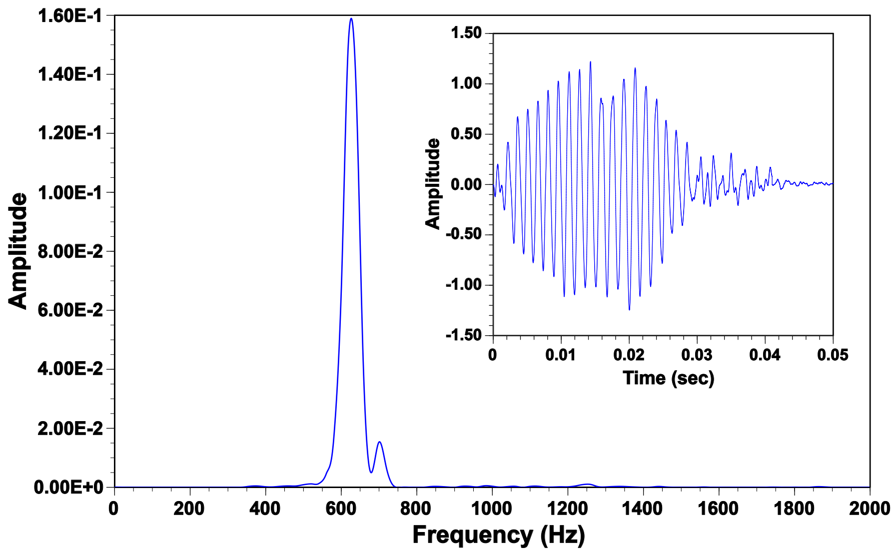


Fig.**9**. same as in Fig. 8 but for the geophone signal. It is remarkable that the two spectra are practically identical in this case.

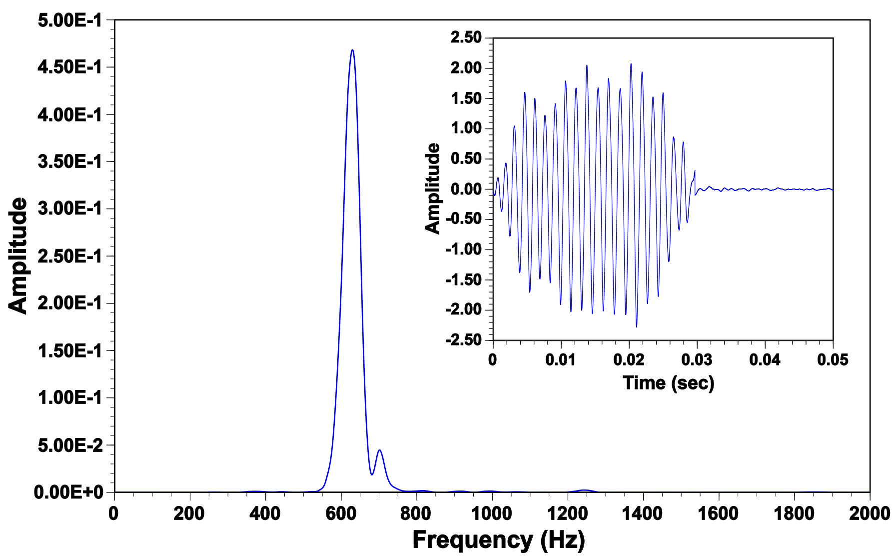


Fig.10. Same as in Fig. 8 but with the sand in a flower pot, with 20 cm rim diameter and 10 cm in depth, nearly one year before the recording of the signal in Fig. 8. One harmonic is clearly seen at $2f_d$ with $f_d \approx$590 Hz.

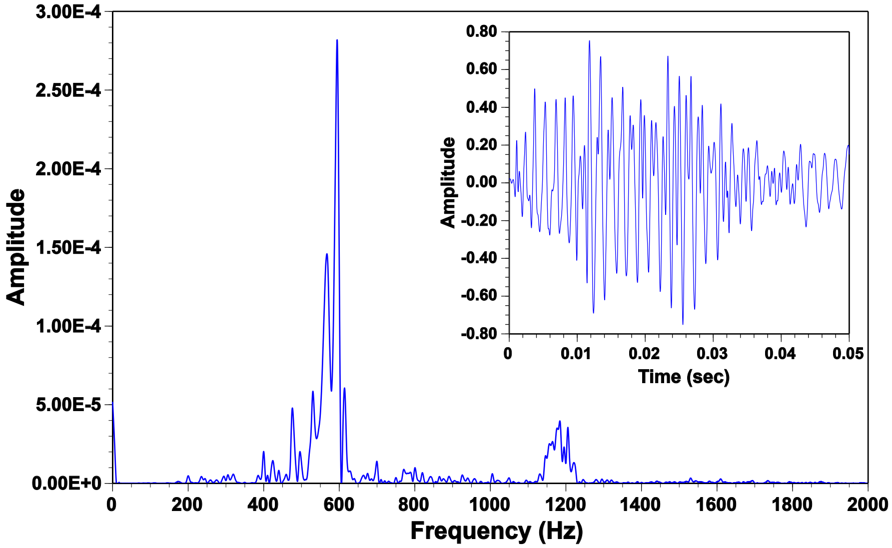
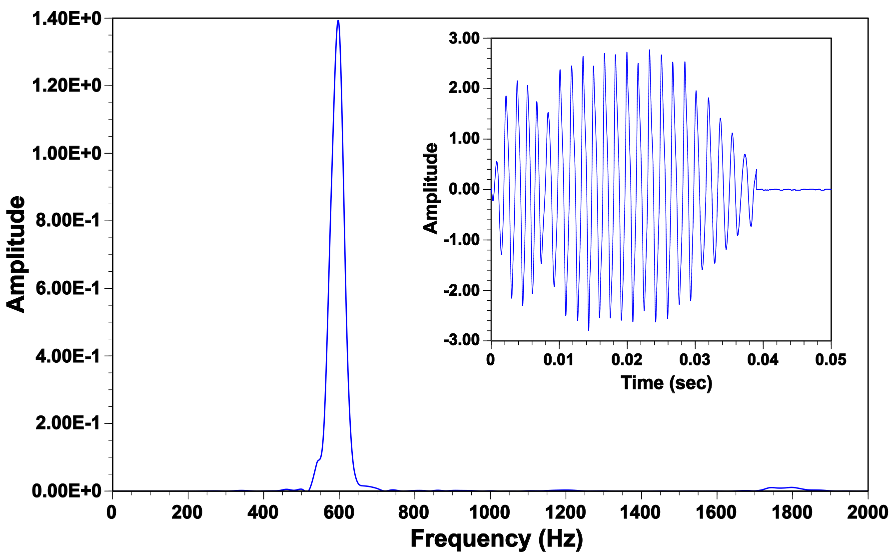


Fig.11. Same as in Fig. 10 but for the geophone signal. The second harmonic is quite discernible at about 1770 Hz with $f_d \approx$590 Hz.



The plots in Figs. 4 to 7 were fairly well reproducible. In these relatively simple geometries, it is safe to argue that the slip shear band is a few mm thick and has the shape of a disk curved around the impacting sphere, with radius roughly that of the sphere [1]. It is also safe to argue that a larger disk radius would imply a larger disk thickness which corresponds proportionately to a lower frequency f_d. From Figs. 4 and 6, the frequency ratio is 578/467=1.24, while the square root of the radii ratio is, 1.25. For the case of the 16 and 25 mm glass spheres, the analogous frequency ratio is, 800/600=1.33. Thus, for these particular cases the slip band thickness increases roughly as the square root of the sphere radius.

The increase of f_d from 590 in Fig. 10 to 630 Hz in Fig. 8, in the course of one year could be due to the aging of the sand. It could also be due to a more or less deviation of the rod axis from the vertical direction. The sand was kept well sealed in plastic bags when not in use. When the same rod was dropped freely and vertically on the same sand bed from the height of about 10 cm, f_d was about 550 Hz, a value definitely lower than 630 Hz when the rod was hand held, Fig. 8. These results lead to the conclusion that the single spring model cannot be applicable in this case, for when hand held, the effective pestle mass is considerably larger than that of the rod and f_d ought to be considerably lower. There is evidence here of the effect of the impact shock on the sand mass when the rod is dropped. It could be argued that it results in longer grain columns and lower f_d.

The absence of the low frequency content, *i.e.*, at around f_p, in all the above plots except for the 11 mm steel sphere, could be attributed to the large energy required for the excitation of the pestle vibration, especially when the pestle is hand held. Thus, when a 16 mm glass sphere was dropped on the same sand bed, in the large plastic container, from the height of about 15 cm, the geophone frequency spectrum included a weak low frequency content at about 70 Hz. In some signals, more than others, the period increases somewhat with time, suggesting that there is a slight increase of the column number N during penetration. The lack of musicality of the low frequency sound emitted, f_d ≈250 Hz, when the flat end of a wood rod, diameter about 8 cm, impacted the Brevort River sand could be attributed to the relatively low flowability of the sand. The duration of the signal amounted to only about 25 ms resulting in only about eight oscillations.

There are only two known reports on harmonics of f_d. In Fig.(c) in [11], f_d=860 Hz and two harmonics are depicted. However, the impaction process of the "squeaking" sand is not specified. In the report by Takahara [24], the singing sand was impacted by a smooth rounded wood rod in a glass funnel. The size of the rod and of the funnel are not specified. The fundamental, f_d=599 Hz, and four harmonics are depicted. A strong first harmonic at

1045 Hz was evoked when the Brevort River sand was impacted by the 7.5 cm glass rod in a regular glass cup with diameters of 8 cm at the rim, 5 cm at the base, 9 cm in height and filled to the height of 7 cm. There were also weaker second and third harmonics.

5.2 Pushed musical grains

Fig.12. The frequency spectrum of the microphone recorded signal when the 7.5 cm glass rod was drawn manually in a nearly vertical position immersed at about 3 cm along the surface of the Brevort River sand in the large plastic container. The signal is shown only up to 160 ms, but the recorded signal lasted up to 500 ms.

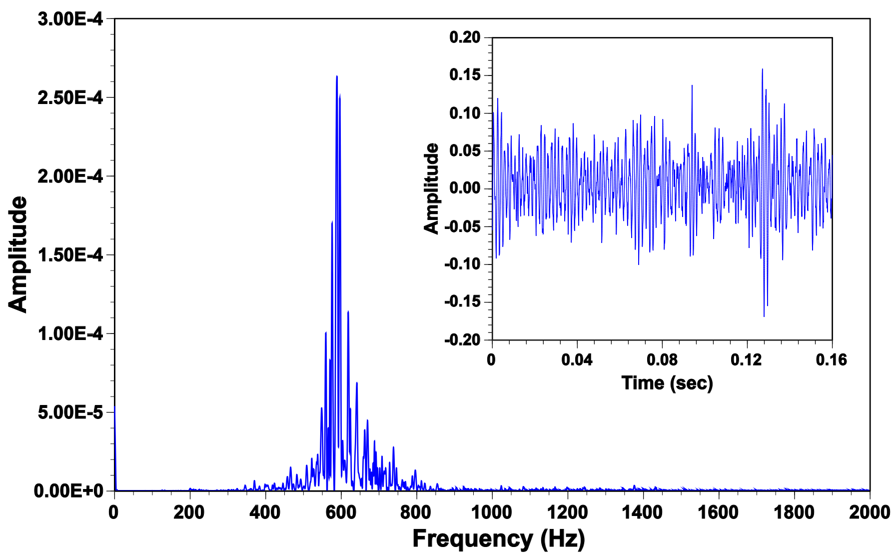


Fig.13. The same as in Fig. 12 but for the geophone recorded signal.

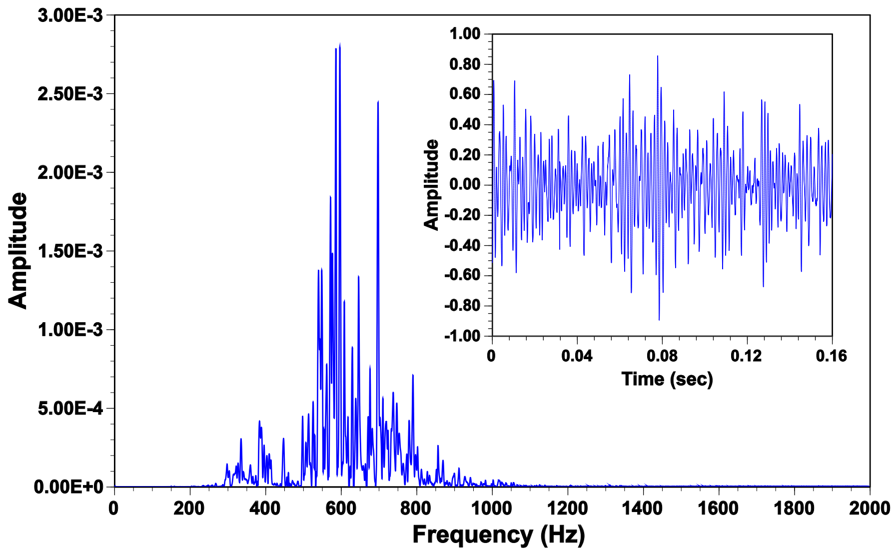
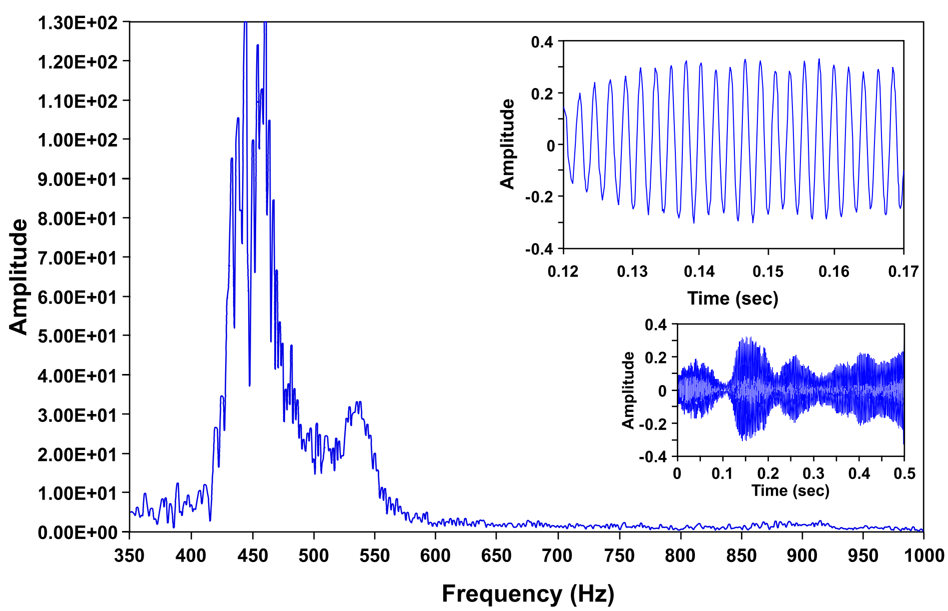


Fig.14. The frequency spectrum of the microphone recorded signal when a 13 mm wood rod was turned manually inside a 9 cm glass jar containing silica gel grains with depth of about 10 cm. $f_d \approx$450 Hz.



The second minor envelope in Fig. 14 at about 540 Hz, could be due to a minor slip band possibly near the bottom of the rod. It could correspond to the sharp peak at about 675 Hz in Fig. 13. The inset signal from 0.12 to 0.17 s happens to be quite monochromatic.

The plots in Figs. 12 and 13 demonstrate that a continuous musical sound, similar to that from an avalanching sand dune, can be emitted by the singing (squeaking) sand grains when sheared (squeezed) in a continuous manner. The spectrum envelope in Fig. 12 resembles quite well that in Fig.(a) in [11], that obtained from the signals found in the website of the author of [2], and that seen in a YouTube presentation by the authors of [4], with the title, Booming Sands, and narrated by Melany Hunt. The ruggedness of the frequency spectrum, as opposed to that in Fig. 6 for example, can be attributed to the continuous renewal of the slip shear band in front of the rod in the course of time. Similar plots were prepared, but not shown, when the same sand was similarly sheared by a 13 mm wood rod. The signals were considerably more noisy and the frequency envelopes considerably more rugged and wider than in Fig. 12. Evidently, the smooth surface texture of the glass rod, as opposed to the wood rod, resulted in considerably more smooth transition from one slip shear band to the next.

5.3 Impacted silent sand grains

Fig.15. Frequency spectrum and the microphone recorded signal when a 13 mm wood rod was tapped (pushed) into a bed of local silent beach sand in the large flower pot, 20 cm at the rim and 10 deep. The weak hissing-like sound was barely audible

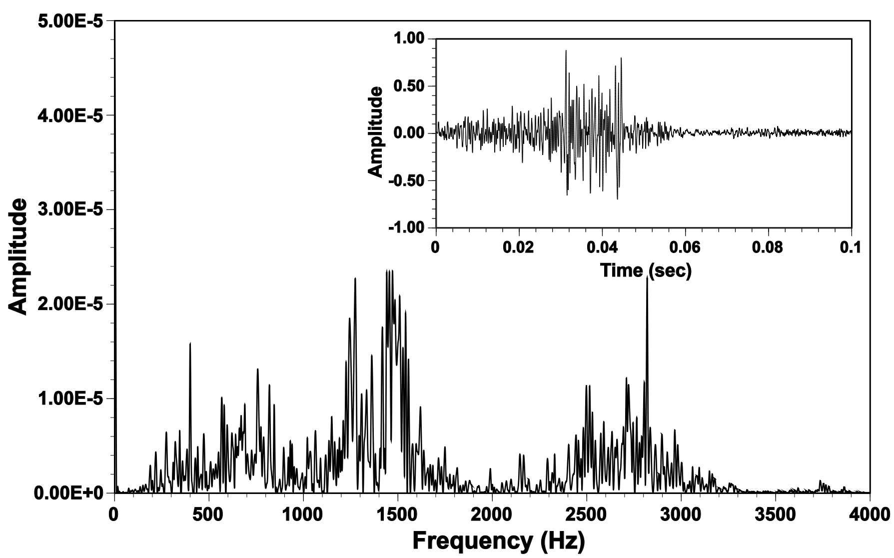
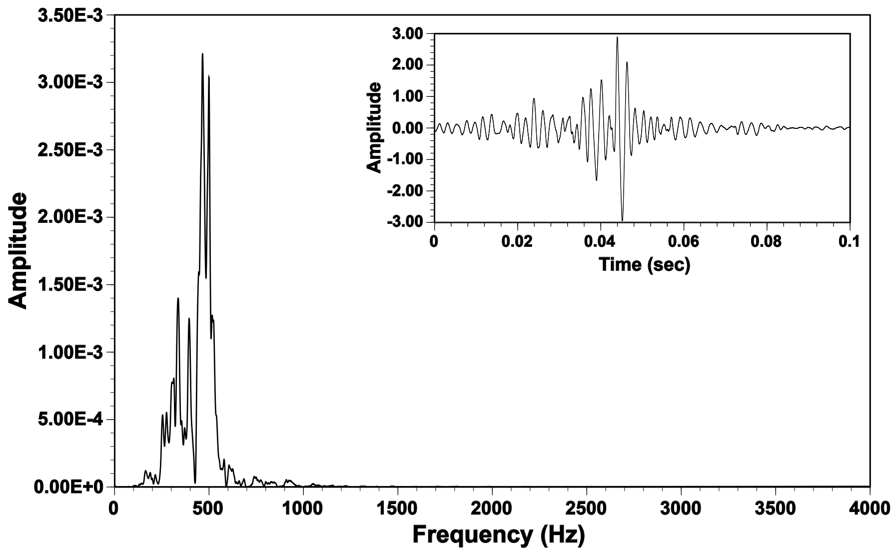


Fig.16. Same as in Fig. 15 but for the geophone recorded signal. $f_d \approx$462 Hz.



The frequency spectrum shown in Fig. 15 is typical of silent sand beds impacted by a variety of pestles. The noise envelope between about 2400 and 3200 Hz is nearly omni-present for all sorts of pestle and grains, suggesting that it originates with grain-grain rubbing near the pestle where the stress forces are maximum. Effectively, these frequencies could correspond to grain asperity collisions as the grains slide past one another. The rest of the noise frequency content could be attributed to grain column vibrations near and parallel to the surface. The grain number N varies randomly and the vibrations are nearly incoherent. This is not the case for the grain columns below the pestle, resulting in a nearly monochromatic vibration in some cases, Fig. 16 for example.

There is considerable variability of the spectrum shown in Fig. 16. In some cases, the envelope is wider comprising many peaks, suggesting that the slip shear band is less well defined, comprising a wider range of grain column numbers N, and in some cases it is narrower without the side peaks between 200 and 400 Hz, for example. However, the noise level approximately above 700 Hz is nearly absent in all cases. It could be argued that the geophone cannot detect efficiently signals in the surrounding medium with wavelength smaller than about its diameter equal to 2 cm. Thus, with V_p=20 m/s, frequencies above 1000 Hz would not be detected efficiently.

Fig.17. Frequency spectrum of the microphone recorded signal when a 13 mm wood rod was tapped (pushed) into a bed of crusher dust in the large flower pot. The grains were very irregular in shape and varied in size from about 1 mm in overall diameter to as large and irregular as $10 \times 5 \times 2$ mm.

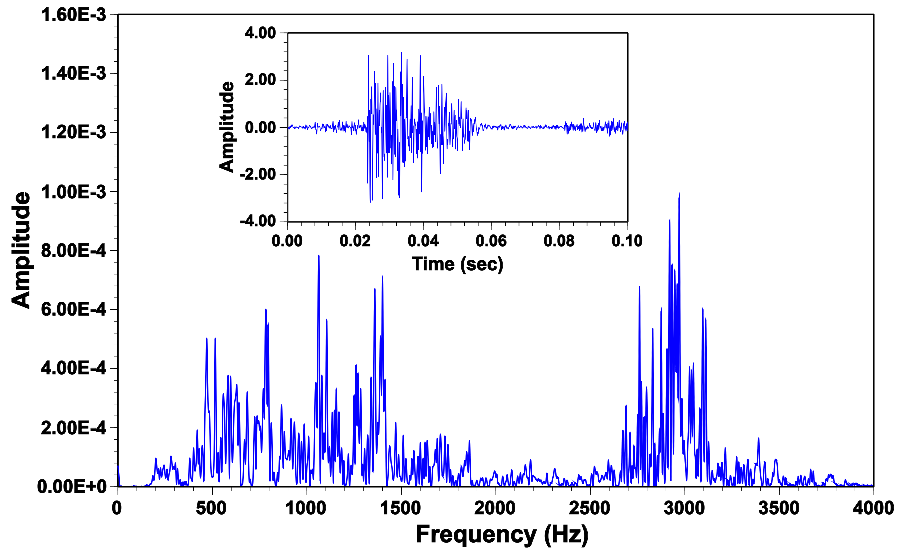


Fig.18. Same as in Fig.17 but for the geophone recorded signal. $f_d \approx$362 Hz.

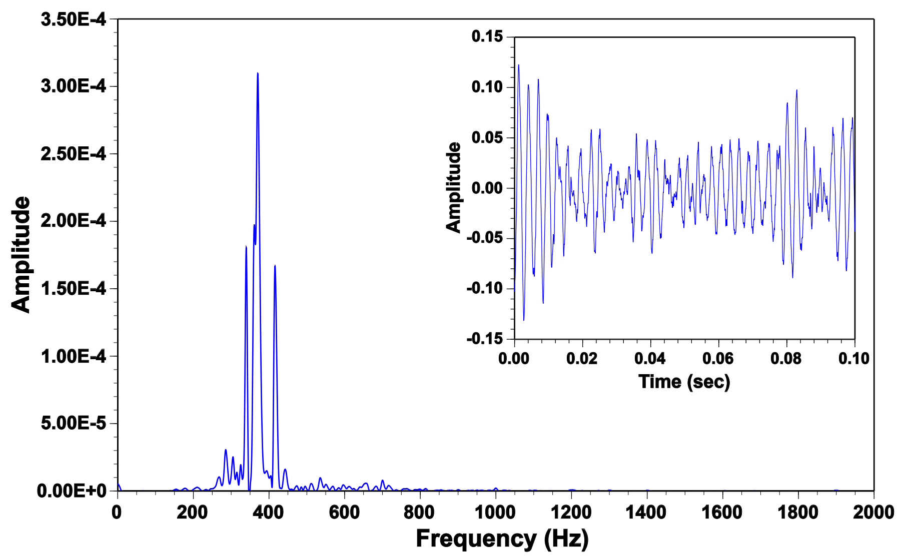
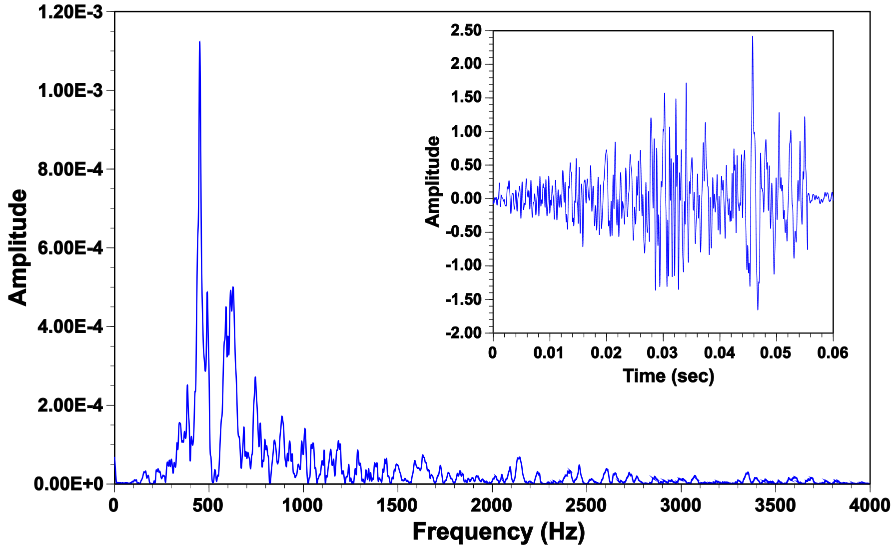


Fig.19. Frequency spectrum of the microphone recorded signal when a metallic rod, 4.2 mm in diameter, was pushed into the same bed of crusher dust in the large flower pot. $f_d \approx 435$ Hz.



Figures 17 and 18 correspond to Figs. 15 and 16 except that the rod was tapped (pushed) into a bed of crusher dust, used as road surface cover in place of pavement. Most of the fine dust had been removed. The use of the relatively small diameter wood rod was necessitated by the large particles in the bed, *i.e.*, a larger diameter rod would not penetrate into the bed smoothly. Despite the unusual crusher dust grain size distribution, the sound was appreciably louder and somewhat more musical than from the silent beach sand. When the same crusher dust bed was impacted by one arm of a walnut cracker, a metallic rod 8 mm in diameter tapered to a smooth rounded end, the microphone noise frequency content around 3000 and 1500 Hz was reduced relatively to that around 500 Hz, while the geophone spectrum was similar to that in Fig. 18.

Furthermore, when the same grain bed was impacted by the nearly blunt end of a walnut scraper, a metallic rod 4.2 mm in diameter with the end polished smoothly for 10 mm and then slightly tapered to a conical end, the emission resulted in a geophone spectrum similar to that in Fig. 18 and in a microphone spectrum shown in Fig. 19. Such a spectrum resembles more closely those of the singing as opposed to those of the silent grains. Evidently, the small diameter and the polished end of the pestle resulted in sufficiently low stress level around the leading front of the pestle so as to reduce the excitation of the grain-grain rubbing and the near surface grain column vibration noise to very low levels.

6. Collective column vibrations and the stick-slip effect

It now appears safe to conclude that the transition from non-singability to singability of a grain bed is based on two premises: (a) the reduction of the stress level on the grains around the pestle and (b) the transfer of the pestle energy to the grain column vibrations below the pestle. The first could be met if there were sufficient grain-grain slippage in the slip shear band below the pestle that would result in low stress level around the pestle, and the second, if the stick-slip effect would be applicable, *i.e.*, if the friction coefficient would decrease with relative velocity between the grains [18].

The degree of grain-grain slippage was tested by dropping a 16 mm steel sphere from a height of 20 cm on grain beds in the 20 cm flower pot composed of: (a) Brevort River sand, (b) Providence Bay, Manitoulin Island, ON, CA, nearly silent sand and (c) a local beach sand sounding even more silent. In case (a), the sphere was barely visible at the center of the crater about 10 mm deep, in case (b), nearly 1/3 of the upper hemisphere was visible and in case (c), nearly all of the upper hemisphere was visible. It is estimated that in case (a), the sphere traveled about twice as far as in case (c) after the initial impact.

Then, there is the question of the synchronization of the column vibrations in the slip shear band. Is it due to a wave propagating in the same band as in the case of the freely avalanching dune sand? In the case of the walnut scraper rod, the diameter of the slip shear band would be equal to about 5 mm, while the synchronization wave would have wavelength equal to 10 cm for propagation velocity and frequency equal to 50 m/s and 500 Hz respectively. Thus, the source of the grain column synchronization has to be sought elsewhere. It could be argued that a few dominant column vibrations would induce the same vibration in the pestle, which in turn would force all columns to vibrate in phase with it, thus, resulting in a collective vibration of all grain columns. In turn, such an intense vibration would facilitate the slipping of the grain mass away from the advancing pestle and would nearly eliminate the surface noise described above. In this sense, the slip band acts like a small monopole radiating shear as opposed to compression waves in the surrounding grain mass, since $\xi_r \gg \xi_z$.

It could be argued that a minimum (threshold) impact pestle velocity is required for the initial excitation of the few column vibrations. Such thresholds are always present before a musical event can occur. When the wood rod was held vertically and forced to move horizontally through a bed of silica gel grains, the immersion depth had to be more than about 2 cm and the velocity had to be more than about 20 cm/s. Moreover, when a plastic bead, 1 cm in diameter, was buried in a flat pile of the Brevort River singing sand and pulled horizontally by a string, the depth had to exceed about 3 cm and the string had to be pulled rather sharply. Similar thresholds are seen in [9] where the booming dune grains were pushed by a blade.

7. Grain flowability and grain confinement

It seems fair to argue that what distinguishes primarily a freely avalanching booming sand, from other silent avalanching sands, is the relatively high avalanche front and the apparent high flowability of the grains. Reference to high flowability, where the sand flow is compared to that of a water stream, can be found in the reports by: Sholtz et al. [25], Bagnold [26] and Humphries [27]. When Brevort River singing sand was placed in a plastic container, 40×35 by 25 cm deep and dumped sharply on the side of a nearby dune ridge with slope over 30^o, there was no appreciable avalanche front. The sand flow was sluggish and characterized more by plate-like motion than free surface grain motion. It was more sluggish than that of ordinary silent sand motion when similarly dumped. However, when a cupful of sand was tossed with some force at an angle of about 45^o on the flat top of a sand pile, the usual sound was evoked.

In [13], it is reported that when booming grains were placed in a sealed glass jar, 17.5 cm in length by 10 cm in diameter, half full, and rapidly tilted, "a violent roar" could be produced. Similarly, when Brevort River singing sand was placed in a glass jar, 17 cm in length by 8 cm in diameter but only 7 cm at the lip, and then tilted sharply, no sound was produced until most of the sand had flown out of the jar and the sand height above the lip was about 15 mm. During the sound emission, the sand appeared to flow out of the jar as in one piece, thus, reinforcing the concept of the slip shear band adjacent to the jar lip.

Furthermore, it is reported in [13] that a hum-like sound was emitted when a 2.5×2.5 cm hole was cut in the cap of the same jar, full of the booming sand, and then it was inverted. In the context of this approach, the grain columns, somewhat above the plane of the hole, would span the hole width, but the grain contact radii, R_b, would have minimum values in the middle of the columns instead of at the top, as in the case of the impacted bed or in the case of the freely avalanching dune sand. Similarly, when booming sand from Sand Mountain in NV. USA, available in the laboratory of the authors of [10] some 20 years ago, was placed in a large plastic funnel, 15 cm rim diameter and 14 mm outlet diameter, a low frequency vibration sound estimated at under 100 Hz could be heard. However, when the same was attempted with the Brevort River singing sand, no sound was evoked. Thus, it could be argued that singing sand grains do not boom because of their low flowability.

Booming dune grains do not sing in the sense of emitting a musical sound when placed in a large dish and impacted by a rod with diameter about 2 cm. In the context of this approach, this is so since the relatively high flowability of such grains results in unstable long grain columns. In other words, the grains can flow away from the rod without the aid of a slip shear band below the pestle. However, when the booming dune surface was impacted sharply by the palm of the hand, there was emission with $f_d \approx 73$ Hz [5]. Evidently, the relatively large area of the impacting hand resulted in a sufficiently large degree of confinement that resulted in a slip shear band a few cm below the hand. Similarly, when booming grains were confined in a 25 cm wide circular channel and pushed by a large blade, they became singable [9]. Additionally, they became singable when confined inside a jar [10]. In [1, 28], it is shown that salt, sugar and silent sand grains can exhibit singability when sufficiently confined. On the other hand, when a flat pile of the Brevort River sand was impacted vertically by the flat end of a wood rod (block) 14 cm in diameter, there was practically no musical sound emission, as was the case for the smaller block described above, due to the relatively low flowability of the grains.

8. The pushed booming sand

During the visit of the authors of [15] to the Sand Mountain, NV, USA, the sand would not boom during free avalanche, but it would emit musical sounds when pushed by the hand or sheared (squeezed) by a shovel blade. Figure. 4 in [15] depicts the frequency spectra of the microphone and geophone recorded signals when a 30×30 cm flat shovel blade was withdrawn sharply with a downward push from the dune face. There are distinct peaks at 67 and 76 Hz in the geophone spectrum, most likely due to distinct slip shear bands under the shovel. In the microphone spectrum, there are several peaks centered at around 60 Hz and a harmonic peak at about 120 Hz, which is also present in Fig. 3_d in [15]. There is also a hint of a harmonic presence in Fig. 5_b and reference to possible low frequency content in the 3 to 30 Hz range, when the sand was dug by the hand.

The first attempt to determine the change in frequency with the manner the sand was pushed on the face of a dune can be found in [13]. It was determined that when the sand was pushed uphill, the frequency increased as opposed to when it was pushed downhill, and it also increased with the speed of push. In Fig. 2 in [26], the "shear plane" is depicted when a sand bed is pushed by a blade and an overburden is heaped up in front of the blade to a height considerably larger than the sand bed depth. The shear plane runs from the foot of the blade to the front of the overburden at an angle, β, from the horizontal bed plane. Intuitively, it could be argued that when the sand is pushed downhill, β tends to zero, and that it tends to larger values when the sand is pushed uphill, resulting in lower overburden mass M_p, and in higher frequency $f_d = f_p$. It is possible, but unlikely that the sound is due to grain column vibrations and that the column number, N, is lower when the sand is pushed uphill. Similar arguments could be used to explain why f_d increases with blade velocity on a horizontal sand bed.

It is claimed in [9] that frequencies, f_d, as low as 25 Hz were obtained by pushing the booming sand on the face of a dune. In the YouTube presentation by the authors of [9], it can be observed that when the sand on the face of a booming dune was squeezed between the two hands, a fairly low frequency sound was emitted. It is highly likely that a slip shear band was formed between the hands and that the pestle mode of vibration was excited with frequency, $f_d = f_p$. It is unlikely that the slip shear band was thick enough so as to result in, $f_d = f_1$. There is no evidence of sand avalanche that would result in such a sound. Again, the question would be resolved when the frequency spectra become available. During such sand squeezing and the subsequent dropping of the sand on the dune surface, the sand appears to respond in the same manner as the Brevort River singing sand except for the lower value of f_d by about a factor of four.

Furthermore, in [9], the study of the change of the dominant frequency, f_d, with blade speed and height of the grain mass in front of the blade is quantified. Booming grains from the Atlantic shores of Morocco were pushed by a 25 cm blade in a cylindrical container, 1 m in dimeter. From Fig. 2 in that report, it can be inferred that for fixed blade velocity, V_b, f_d varies nearly as the inverse of $\sqrt{H_b}$, where H_b is the sand height in front of the blade, implying that $f_d = f_p$. However, the increased sand height could result in thicker slip shear band and in $f_d = f_1$. The ambiguity would be resolved when the spectra of the emitted signals are available.

9. Freely avalanching booming sand

When a sand band, several cm thick, is in a state of avalanche, it is effectively confined by the plane of the static sand below and the thin band of about 20 relatively fast moving surface layers above [2, 9]. It can now be argued that the grain layers near the bottom of the avalanche front experience the greatest stress level when they decelerate sharply and are overtaken by the layers above, and that a slip shear band could exist in that region. It is possible that the so called "roar sound" emitted when the sand is pushed downhill in a heaped-up manner, [13], is due to grain column vibrations in such a front slip shear band. If there were no harmonics of f_d, then, such a sound would be due to the overburden vibration with frequency $f_d = f_p$. However, the "hum" that follows the "roar" represents a steady state acoustic emission that is independent of an avalanche front. It can be maintained by continuously digging a hole where the avalanching sand is deposited [13]. It is reported in [2, 9] that the minimum thickness of an avalanche for booming to occur is about 2 cm, implying grain column number, $N \geq$100, with $\bar{d}$=0.18 mm.

Figure 5 in [13] depicts the ripples (waves) on the dune surface when the sand was pushed downhill in a heaped-up manner. There are several avalanche fronts followed by flat plateau-like segments. The author describes the sound as, "being due to the hum accompanying the roar". In the context of this approach, the roar would be due to column vibrations in the front slip shear bands, while the hum would be due to similar vibrations in the surface of the flat segments, as described below. The column vibrations in the entire avalanche area would be synchronized by the surface modes of wave propagation excited by the avalanching grains, [2, 5-7].

Before proceeding with the usual computations of the eigenfrequencies, f_p, f_1,.. in the grain columns, it is deemed appropriate to try to elucidate the relevant experimental information available.

(a). During a free booming avalanche, the frequency spectrum consists of the main envelope centered at the dominant frequency, f_d, harmonics of f_d and the a low frequency envelope centered at the frequency $f_p \approx 1/3 f_d$. Thus, on the basis of the geophone signals obtained on

dunes in Atlantic Sahara, Morroco, by the author of [2], $f_d \approx$100 Hz and $f_p \approx$ 30 Hz. Permission to this effect was obtained from the author of [2]. Furthermore, a similarly structured spectrum can be seen in a YouTube presentation of the authors of [4] with the title, Booming Sands, and also one by the authors of [9] with the title, Song of Dunes.
(b). The propagation velocity, V_p, of the surface waves generated during a booming avalanche, or by a speaker in a static dune sand, was measured at about 40 m/s, [1].
(c). On page 3 in [4], it is stated that: "By cross-correlating the geophone signals, the phase speed of booming is measured at 200 m/s near the crest of the dune and increasing to 350 m/s further downhill. Hence, booming results from the propagation of body waves not surface waves." However, on top of the same page, it is stated that surface waves propagate with speed of about 50 m/s and are strongly attenuated. Furthermore, from the discussion on page 1 in [4], it could be inferred that the booming sounds were due to forced slides as opposed to freely avalanching dune sand, implying that f_d does not have to scale as $\bar{d}^{-0.5}$ as reported in [2] for freely avalanching sand.

In this sense, the question arises as the whether the geophones on the dune surface were detecting signals from the surface waves generated by the forced avalanche ahead of the slide agency, or signals from the "body" compression waves generated by the slide agency. The authors do not elaborate on the increase of V_p with depth, but according to [16], the bulk of such "shot" waves would not travel along the dune surface. It could be argued that they would travel along the bottom of the surficial layer at the depth of about 2 m, where the propagation velocity would be about 200 m/s. In Fig. 5 in [16], V_p in unconsolidated beach sand increases, nonlinearly, from about 40 to 160 m/s when the shot-receiver distance increases from 0.1 to 1.5 m, implying that the shot waves do not propagate along the sand surface.

It is possible to obtain a rough estimate of the compression phase velocity, c_p, in the slowly avalanching slip shear band, by the following reasoning: On the basis of (b) above, V_p would assume the value, V_p=50 m/s at the depth ζ= 5 cm. Then, at ζ=200 cm, V_p would be equal to 92 m/s, or 124 m/s or 170 m/s, if V_p scaled as $\zeta^{1/6}$, or $\zeta^{1/4}$, or $\zeta^{1/3}$ respectively, the last case resulting in V_p closest to 200 m/s. Since the confining hydrostatic pressure is, $P = \rho 9.8\zeta$, V_p scales with P as with ζ. The need for reconsideration of the $P^{1/6}$ scaling seems to have been raised earlier by Goddard [29], where a scaling as $P^{1/4}$ is discussed. Then, in Fig. 5 in Makse et al. [30], it is seen that both elastic moduli in the grain mass scale as P^{δ} with δ somewhat larger than 1/3 and in Fig. 9, using molecular dynamics simulations, the shear modulus for frictionless grains scales as $P^{2/3}$ implying that V_s scales as $P^{1/3}$. Then, in Fig. 8 in Bachrach et al. [17], the shear velocity, V_s, in the range, 100 to 175 m/s appears to scale as $\zeta^{1/3}$ in the range 2 to 10 m.

It is now assumed that the compression phase velocity, c_p, in the contact shear bands, is equal to V_p and since V_p increases slowly with ζ in the small interval, $0 < \zeta < 2$ cm, c_p is assumed equal to 38 m/s, *i.e.*, equal to V_p in the middle of the avalanching band at $\zeta \approx$1 cm. The value, V_p=38 m/s was arrived at by assuming that V_p scales as $\zeta^{1/3}$ and that V_p=50 m/s at ζ=5 cm, on the basis of information in (a, b) above.

In the opening paragraphs of Section 2, it is shown how to determine the value of $\bar{d}/R_b$ when two ideally spherical and homogeneous grains are pressed together by the force, F, namely, $R_b = (C_1/C_2)^{1/3}$, where, $C_1 = 3FR(1-\nu)$ and $C_2 = 8G$, where, G, ν are the shear modulus and the Poisson's ratio of the grains. Then, with $\bar{d} = 2R$=0.18 mm, ν=0.2 and $G = 4 \times 10^{10}$, $\bar{d}/R_b$=1640 at the surface of the slowly avalanching shear band and $\bar{d}/R_b$=1100 at the depth $\zeta = 48\bar{d}$=0.86 cm, where, F is equal to the weight of 20 and 68 grains respectively. Evidently, R_b scales as $\zeta^{1/3}$ since F scales as ζ.

The available software could not process matrices larger than about 50×50 and so N was limited to 48. Then, with $\bar{d}/b$=4000, c_p=38 m/s and $\bar{d}/R_b$ decreasing with depth from 1640 to 1100, the spectrum was computed as twice the following values, f_p=39 Hz, f_1=100 Hz, f_2=186 Hz etc. If R_b would not increase with depth, then, $N = 2 \times 48$=96 would result in the above values for f_i. But, since R_b increases rather weakly with depth, as $\zeta^{1/3}$, a somewhat larger N, *i.e.*, $N \approx 3 \times 48 \approx 134$, resulting in column length equal to $134\bar{d}$= 2.4 cm, would result in the frequency values listed above.

It may be noted that the particle displacement, ξ_r, along the direction of slide, for the mode corresponding to f_1, is characterized by a node at about the center of the column, (1, 2) in Appendix A. This is in agreement with the experimental data depicted Fig. 3d and the discussion that follows on page 4 in [7], where the authors claim that the particle (grain) displacement, along the avalanche direction, has a node near the center of the avalanching band.

It is worthy of note that for N=96, the average value of R_b at the middle of the avalanching band at N=48 is equal to 0.165×10^{-3} mm, while that of R_b under the 11 mm steel sphere impacting the Brevort sand is equal to 1.30×10^{-3} mm, resulting in R_b ratio equal to 7.9, *i.e.*, equal to the corresponding dominant frequency ratio in the the two cases.

According to equation (9) in Appendix B, V_c and by extension V_p could be expected to scale as R_b and thus as $\zeta^{1/3}$, a result of the assumption of the existence of the contact shear bands. In other words, the stiffness of the grain columns that results in (9) varies as R_b^2, while, according to the Hertz-Mindlin theory it would vary as R_b, as seen in (1, 2) in [17]. However, it must be remarked that (9) was based only on compressive vibrations in a grain column. Furthermore, in Fig. 6 in [16], V_p for beach sand, derived from the Hertz-Mindlin theory, exceeds the measured values by a factor of 2.

During a silent avalanche on a dune surface or in the initial stage of an avalanche on a boomable sand dune, a given grain, in one of the 20 or so fast avalanching layers, generates a quasi-periodic elastic wave train composed of the impact pulses upon collisions with the grains in the layer below. The average time between collisions, T_c, is given in the first paragraph in the Introduction. The chance that the given grain will collide with a grain below is maximum when the grain below collides with another grain in the layer further below, since then it is nearly stationary. Furthermore, the kinetic behavior of the grains on either side would be influenced due to the close grain proximity. Thus, it can be argued that the various quasi-periodic wave trains are somewhat synchronized and the resultant wave amplitude is sufficiently large to initiate the excitation of the column vibrations in the slowly avalanching grain band.

In this sense, the column vibrations, within a radius of several grain diameters, are nearly in phase, and if the conditions for the applicability of the stick-slip effect exist, the resultant amplitude of the column vibrations would be sufficient to generate the surface waves that would synchronize all column vibrations within areas as large as several m^2 as reported in [2]. When such areas of synchronization are repeated several times over a large dune surface and the avalanching band extends to several cm in depth, the seismic and acoustic energy output would be enormous, resulting in difficulty in standing up nearby, and in audible sound up to 2 km away [11, 31]. Then, the question arises as to whether a geophone buried a few cm below a silent avalanche would record a fairly monochromatic signal, as in the case of the impacted silent grains.

The collective grain column vibrations cannot be sustained at frequencies other than the average grain-grain collision rate, f_c $\approx$100 Hz, in the fast avalanching surface layers, for $\bar{d}$ $\approx$0.18 mm. Unlike the case of the singing grains where the energy that excites the column vibrations is derived from the impacting pestle, in this case the energy is derived partly from the grain-grain collisions in the same surface layers. Thus, any collective vibration has to be slaved to the frequency f_c. However, the frequency f_1 is defined by the stiffness of the contact shear bands, and if f_1 is appreciably different from f_c, then, boomability is not possible. Such a conclusion is consistent with the rarity of such phenomena. They occur only during certain periods of the year and not all dunes in a given region can boom during a free avalanche, even though, they can produce the booming sound when squeezed by a shovel blade [15] or pushed by the hand or a blade [9]. Furthermore, if the thickness of the avalanching sand band becomes too thin, then, f_1 is forced to exceed f_c and boomability ceases, as reported in [2, 9].

10. Sliding booming sand plates

Along the path of many suggestions and assumptions during the development of this study, not always correct, it seems fair to assume that there is a slip shear band (column shear band), in front of the sliding plate, responsible for the seismic and acoustic emissions. How far such a channel extends below the plate remains an open question. The channel width would be rather thin given the chaotic dynamics of the grain motion under the plate. Thus, the relatively low value of f_d would imply the single spring mode of vibration where the relatively large overburden mass, M_p, would be responsible for the low value of $f_d = f_p$.

As in the case of the impacted grains, the gravitational energy of the plate can be transferred to the column vibrations via the stick-slip effect. At this stage, it is deemed appropriate to include an excerpt from the book by Curzon [31], p. 285, that appears to correspond to the observation by Vriend et al. [4] in that dune vibrations were detected even when there was no apparent avalanche in progress. "By the flowing in of the sand from the sides and the repeated tread [of the traveler] a large part of the whole sand-layer of the slope at last acquires motion, and by its friction against the motionless under-layer produces a noise, which from a humming becomes a murmur, and in the end passes into a roar, and is all the more surprising in that one sees but little of the trickling and general movement of the sand-layer."

11. Conclusions

It is possible to formulate a theoretical model, based on the existence of a number of discrete grain contact shear bands forming grain columns, that could account for the experimental evidence on the behavior of singing grains, particularly the frequency spectra of the signals emitted when the grains are impacted by a pestle. Thus, the dominant frequency, f_d, and its harmonics correspond to the modes of vibration in the grain columns below the pestle with fundamental frequency f_1 and harmonics of f_1. The low frequency content, corresponds to the pestle mode of vibration where the grain columns act as short single springs. The conversion of the pestle kinetic energy into elastic energy of such modes is effected by the stick-slip effect in the contact shear bands, in agreement with the widely held viewpoint that grain singabilty depends critically on the grain surface state. As the grains slide over one another, the pestle energy is transferred to particle oscillations along the slide direction, in the contact shear bands, and from there to column vibrations. Additionally, singability is independent of grain shape and grain size distribution.

The contact shear bands play a similar role to that of the water layer when a finger is drawn on a wet glass surface resulting in the usual squeal sound. In this sense, the singability of an impacted grain bed is the result of sufficient grain-grain slippage between the grains in a given grain column. The pestle forces the grain columns to vibrate in phase with it, resulting in the collective vibration of all columns. The ensuing intense vibration below the pestle facilitates the slipping of the grain mass away from the pestle and nearly eliminates the surface noise generated by the intense grain-grain rubbing and the chaotic grain column vibrations adjacent to the pestle near the surface.

Similarly to the impacting pestle, in the case of the freely avalanching dune sand, the dominant frequency, f_d, and its harmonics correspond to the modes of vibration in the grain columns, in the slowly avalanching sand band, with fundamental frequency f_1 and harmonics of f_1. The low frequency content, at $f_p \approx 1/3 f_d$, corresponds to the pestle (overburden) mode of vibration where the grain columns act as single short springs. Very low frequency emissions, at around 30 Hz, occuring when booming sand is pushed or squeezed on a dune surface, could be due to the same pestle mode of vibration. Acoustic and seismic emissions from seemingly motionless dune sand, following an avalanche, could be due to minute sand plate motion.

The large variation in the values of the measured wave velocities in sand grain beds could be attributed to the large variation in the size of the grain contact shear bands, due in part to a variable grain surface coating.

Singing grains do not boom when forced to free-avalanche since their relatively low flowability does not allow for an orderly surface flow comprising a fast avalanching thin band and a slower well defined band below, several cm thick. Similarly, booming sand grains do not sing when impacted in a dish by a pestle since their relatively high flowability results in sufficient sand motion away from the pestle without the need for an intense vibration in the slip shear band below the pestle. The conversion of gravitational into grain column vibration energy is effected by the stick-slip effect. In this sense, ordinary sand grains do not boom during an avalanche since the stick-slip effect is not applicable in the grain contact areas. The assumed grain contact shear bands result in wave propagation velocity scaling with sand depth, ζ, as $\zeta^{1/3}$ instead of $\zeta^{1/6}$ predicted by the Hertz-Mindlin contact theory.

The extreme sensitivity of the boomability of a given sand dune to weather and other factors could be attributed to the sensitive dependence of the fundamental frequency f_1 of the column vibrations in the avalanching band on such factors. That is, if $f_1 \neq f_c$, boomability cannot occur, where, $f_c = 1/T_c$ and T_c is the average time required for one grain to overtake another in the fast avalanching surface band of about 20 layers. In this sense, boomability requires a fairly narrow grain size distribution. Furthermore, if the avalanching band is too thin, then, f_1 becomes greater than f_c and boomability is not possible.

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Appendix A. Modes of vibration in a grain column

In what follows, the origin, O in Fig. 1, is assumed to coincide with the center of shear band # 1 and the direction along $\hat{x}$ coincides with that along $\hat{r}$ at θ=0. It is more instructive to use cylindrical instead of Cartesian coordinates to derive the relevant equations. The particle displacement, ξ_s, is written as, $\xi_{\mathbf{s}} = \nabla \times \mathbf{A}$, where $\mathbf{A}$ satisfies the vector wave equation with phase velocity c_s. $\mathbf{A}$ is chosen to lie along $\hat{\theta}$ resulting in, $A_\theta = [A_1 \text{cos}\alpha z + B_1 \text{sin}\alpha z] J_1(\beta r) \text{e}^{j\omega t}$, where, $J_1(\beta r)$ is the Bessel function of order 1 and, $\alpha^2 + \beta^2 = k_s^2 = (\omega/c_s)^2$. This in turn results in,

$$\xi_z = [A_1 \text{cos}\alpha z + B_1 \text{sin}\alpha z]\beta J_1'(\beta r) \approx [A_1 \text{cos}\alpha z + B_1 \text{sin}\alpha z]\beta \text{cos}\beta \text{r} \tag{1}$$

and

$$\xi_r = [A_1 \text{sin}\alpha z - B_1 \text{cos}\alpha z]\alpha J_1(\beta r) \approx [A_1 \text{sin}\alpha z - B_1 \text{cos}\alpha z]\alpha \text{sin}\beta \text{r} \tag{2}$$

where, prime indicates derivative with respect to the argument and the factor $\text{e}^{j\omega t}$ is understood to be included. On physical grounds, ξ_z cannot depend on the polar angle θ, and thus it is assumed that $\partial A_\theta/\partial\theta$=0. In particular, ξ_z cannot include the factor cosθ since it cannot change sign as θ is varied from 0 to 2π.

In shear band #1, the expression for the particle displacement along $\hat{z}$ simplifies to,

$$\xi_{1z} = [B_1 \text{sin}\alpha z]\beta J_1'(\beta r) \tag{3}$$

assuming that ξ_{1z}=0 at z=0, and that in shear band # 2 becomes,

$$\xi_{2z} = [A_2 \text{cos}\alpha z + B_2 \text{sin}\alpha z]\beta J_1'(\beta r) \tag{4}$$

In shear band #3, the coefficient subscripts become 3 and so on for the rest of the bands.

The boundary condition on the top of shear band # 1 at $z = b$ is,

$$\int \sigma_{1zz} r dr d\theta + \int \sigma_{2zz} r dr d\theta = M_1 \partial^2 \xi_{1z}/\partial t^2 \tag{5}$$

where M_1 is the mass of grain # 1 and $\partial^2 \xi_{1z}/\partial t^2$ is evaluated at r=0. The normal stress per unit area along $\hat{z}$ is given as, $\sigma_{1zz} = -(\lambda_e + 2\mu_e)\partial\xi_{1z}/\partial z$ =-$\rho c_p^2 \partial\xi_{1z}/\partial z$, while that along -$\hat{z}$ at the bottom of shear band # 2 is given as, $\sigma_{2zz} = \rho c_p^2 \partial\xi_{2z}/\partial z$. The mass density in the bands was assumed to be equal to that in the grains, *i.e.*, that of quartz equal to 2650 kg/m^3. Equation (5) is repeated at the top of shear band # 2 until the top of the last band # 6, where the normal shear force σ_{6zz} acts on the equivalent pestle mass M_p, *i.e.*,

$$\int \sigma_{6zz} r dr d\theta = M_p \partial^2 \xi_{6z}/\partial t^2 \tag{6}$$

The result of (5), with A_1=0, is the following working equation,

$$[-S_1 \rho c_p^2 \alpha \cos(\alpha b) + M_1 \omega^2 \sin(\alpha b)]B_1 - [S_2 \rho c_p^2 \alpha \sin(\alpha_1)]A_2 + [S_2 \rho c_p^2 \alpha \cos(\alpha_1)]B_2 = 0 \tag{7}$$

where S_1, S_2 are the areas of bands # 1 and 2, $\alpha_1 = \alpha(b + d_1)$, and ξ_z is practically constant over the contact area since cos$\beta r \approx$1.

The grains are assumed to be perfectly rigid, so that $\xi_{1z}(z = b) = \xi_{2z}(z = b+d_1)$, resulting in the working equation,

$$\sin(\alpha b)B_1 - \cos(\alpha(b + d_1))A_2 - \sin(\alpha(b + d_1))B_2 = 0 \tag{8}$$

where d_1 is the overall diameter of grain#1. There are $2N + 1$ equations and $2N + 1$ coefficients, $B_1, A_2, B_2, ...A_6, B_6$ for N=5 grains as in Fig. 1.

The eigenfrequencies, $f_1, f_2, ..f_N$, are determined by looking for the zeros of the determinant of the coefficients, B_1, A_2,.. when $\omega = 2\pi f$ is varied, provided the wavenumber α can be specified in terms of ω. Then, for a given eigenfrequency, the coefficients can be specified relative to the arbitrary value of B_1=1, and the nature of the corresponding mode of vibration can be examined. However, the value of the wavenumber β must be specified before α can be specified from the relation, $(\omega/c_s)^2 = \alpha^2 + \beta^2$. However, the frequencies of the column modes of vibration are defined by the rate of change of ξ_z with z, (5), *i.e.*, the frequencies do not depend on β, as can be seen in (7), and this implies that $\beta \rightarrow$0. A measure of the magnitude of β in terms of α may be reasonably expressed as, $\beta R_b = \alpha b$, where, $R_b/b \approx$25.

Appendix B. Propagation velocity in a grain column

An attempt is made here to derive an expression for the propagation velocity in a column (chain) of grains lying on a frictionless horizontal floor in order to emphasize the irrelevance of the force of gravity. The grains are assumed to be spherical and identical with diameter, d, mass, m, and the contact shear bands with thickness, b, are replaced by identical short springs of spring constant, k. If an equivalent Young's modulus for the thin contact bands is defined as, $Y_b = F/(\delta z/b)$, then, $Y_b = kb$. On page 305 in Symon [1], the problem of wave propagation in a string (chain) of point particles of mass m, interparticle distance, h, and string tension, τ, is treated. A similar procedure can be used in this case by writing the force equation for the n^{th} grain as, $md^2u_n/dt^2 = k(u_{n-1} - u_n) - k(u_n - u_{n-1})$ where the displacement of the n^{th} grain is, $u_n \approx nd$, since $b \ll d$. Then, by introducing the mass per unit length, $\sigma = m/d$, the wave equation can be obtained, with phase velocity, $V_c = \sqrt{kd/\sigma} = \sqrt{(d/b)(Y_b/\sigma)}$. Furthermore, if the compression phase velocity in the contact bands is defined as, $c_p = \gamma\sqrt{(Y_b)(/\sigma_b)}$, then, $V_c/c_p = \sqrt{(d/b)(\sigma_b/\sigma)}$, where σ_b is the mass per unit length in the contact bands and γ is a correction factor. It can be shown that, $\sigma_b/\sigma = 3/2(R_b/R)^2$, where, R_b, R are the radii of the contact bands and the grains respectively, with the assumption that the mass densities in the grains and the contact bands are the same. Thus,

$$V_c/c_p = (1/\gamma)\sqrt{3}(R_b/R)\sqrt{R/b} \tag{9}$$

It is worthy of note that the equivalent spring constant, k, varies as R_b^2 and as $1/b$ and thus, V_c varies as $R_b\sqrt{1/b}$.

References

1. K. R. Symon. Mechanics, 3rd edition. Addison-Wesley, Reading, MA, USA, 1971.